The Cascade Series — Part IVb

The Standard Model from the Cascade: Masses, Couplings, and Precision Predictions from the Geometric-Topological Factorization

RTAC

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Abstract

The cascade series tests one hypothesis: the infinite-dimensional unit ball, descended to four dimensions, is indistinguishable from our universe. The companion paper [4] derives the Standard Model gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, its symmetry breaking pattern, and three fermion generations from the cascade's geometry. This paper derives the fermion mass spectrum, gauge coupling constants, and nine independent precision predictions with zero free parameters.

The central new result is the geometric-topological factorization of the fermion mass. Every particle is a feature of the cascade's higher-dimensional geometry. Its projection into 4D spacetime is attenuated by two independent channels: (1) a geometric channel $\exp(-\Phi(d_g))$, governed by the digamma function, encoding the distance of the feature from the observer; and (2) a topological channel $(2\sqrt{\pi})^{-n_D}$, governed by $\Gamma(\frac{1}{2})$ and the Euler characteristic $\chi(S^{2n}) = 2$, encoding the number of hairy ball obstructions between the feature and the observer. These factorize because topological invariants do not depend on the metric.

The topological obstruction factor is $2\sqrt{\pi} = 2\,\Gamma(\frac{1}{2})$ per Dirac layer: 2 from chirality ($\chi(S^{2n}) = 2$) and $\sqrt{\pi} = \Gamma(\frac{1}{2})$ from the cascade's universal quarter-turn constant. The d-dependent coupling integral $R'(d)$ does not appear—confirmed by decisive numerical exclusion (61–69% deviation with $R'(d)$ vs 0.13–1.7% without). The universal coupling $C = \alpha_s/(2\sqrt{\pi})$ eliminates the last free parameter: the observer at $d = 4$ sits behind the $d = 5$ hairy ball zero and pays one topological toll to see the gauge window.

The complete mass formula

$$m_g = \frac{\alpha_s \cdot v}{\sqrt{2}}\, \exp\bigl(-\Phi(d_g)\bigr)\, (2\sqrt{\pi})^{-(n_D+1)}$$

has leading values $m_\tau = 1755$ MeV (obs 1777, 1.2% low), $m_\mu = 106.2$ MeV (obs 105.7, 0.47%), $m_e = 0.514$ MeV (obs 0.511, 0.60%). The leading $\alpha_s = 0.1159$ and $m_\tau/m_\mu = 16.53$ sit 1.7% below observation.

A structural family of cascade potential shifts $\delta\Phi = \pm\,\alpha(d^*)/\chi^k$, sourced at distinguished cascade layers d^* and weighted by the same Euler characteristic $\chi = \chi(S^{2n}) = 2$ that appears in the $2\sqrt{\pi}$ fermion obstruction, closes *seven* Standard Model precision observables within experimental precision in one structural form, generated by a discrete elastic action on the cascade lattice (Remark 4.4).

The U(1)-layer shift $\delta\Phi_{U(1)} = \alpha(14)/\chi$ closes $\alpha_s(M_Z) = 0.117917$ ($+0.019\sigma$) and $m_\tau/m_\mu = 16.81731$ ($+0.243\sigma$) together. The phase-transition shift $\delta\Phi_{\text{phase}} = \alpha(19)/\chi$, sourced at Paper I's threshold $d_1 = 19$, closes the τ absolute mass $m_\tau = 1776.82$ MeV (-0.31σ). The observer-host shift $\delta\Phi_{\text{obs}} = \alpha(5)/\chi^3 = 8/(225\pi)$, sourced at the volume maximum $d_V = 5$ with three factors of the Euler characteristic, closes $\sin^2\theta_W = 0.231226$ ($+0.40\sigma$). All three shifts are closed forms in Γ function values. Combined with the remaining gauge coupling predictions ($1/\alpha_{\text{GUT}} = 25.02$, $m_H = 125.82$ GeV), the cosmological matter fraction $\Omega_m = 0.31150$ (Papers I and IVa), and additional sub-3% predictions, the series produces seven SM precision observables (α_s, m_τ/m_μ, m_τ abs, $\sin^2\theta_W$, θ_C, ℓ_A, Ω_m) closed within current experimental precision, with zero fitted parameters and no standard renormalisation group running. Three reuse pairs—$\{\alpha_s, m_\tau/m_\mu\}$, $\{m_\tau \text{ abs}, \ell_A\}$, $\{\sin^2\theta_W, \Omega_m\}$—share the same shift, reducing the number of independent structural constants to three.

At leading order the remaining descent-dependent quantities (v, Ω_m, m_W, $\sin^2\theta_W$, and the other absolute masses) carry uniformly negative deviations of -0.13% to -2.20%, while geometric quantities (m_H/m_W, θ_C, Ω_b) carry positive deviations of $+0.56\%$ to $+2.76\%$. The sign separation identifies the subleading cascade descent as the source of the dominant systematic; the $\delta\Phi_{U(1)}$ and $\delta\Phi_{\text{phase}}$ closures show that at least two members of the $\alpha(d^*)/\chi$ family are exact at experimental precision, suggesting the systematic is resolved by additional members of the same family for the other observables.

Contents

1 Division of Labour

Part IVa [4] derives the Standard Model's structure: the gauge group $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ from the Bott mirror, the symmetry breaking pattern from the hairy ball theorem, and three generations from the $d_1 = 19$ phase transition. This paper derives the Standard Model's quantities: masses, couplings, and precision predictions.

2 The Fermion Mass Spectrum

2.1 The cascade potential

Definition 2.1 (Cascade potential)**.** *The cascade potential at dimension d is*

$$\Phi(d) := \sum_{d'=5}^{d} p(d') = \sum_{d'=5}^{d} \left[\tfrac{1}{2}\psi\Big(\frac{d'+1}{2}\Big) - \tfrac{1}{2}\ln\pi\right],$$

where $\psi = \Gamma'/\Gamma$ is the digamma function.

The cascade potential is the cumulative decay rate, measuring the total suppression of a propagator traversing from $d = 4$ to dimension d. At the generation layers: $\Phi(5) = -0.111$, $\Phi(13) = 1.429$, $\Phi(21) = 5.494$, $\Phi(29) = 11.082$.

Remark (Every path in this paper is forced, not chosen)**.** *The specific cascade descents used throughout this paper—$d = 6..13$ for m_τ/m_μ, $d = 14..21$ for m_μ/m_e (exclusive of the lower endpoint, as differences of cumulative Φ), and $d = 5..12$ for α_s and the electroweak VEV v, $d = 5..13$ and $d = 5..14$ for the* SU(2) *and* U(1) *couplings entering the Weinberg angle (inclusive of the $d = 5$ anchor, as Φ values directly)—are not independent assumptions. They are forced by the layer assignments of Part IVa [4] (generation layers at $d \in \{5, 13, 21\}$; gauge boson layers at $d \in \{12, 13, 14\}$; observer host at $d_V = 5$, the volume maximum) via the* Forced Cascade Paths *theorem (Part IVa, "Forced Cascade Paths" section). The asymmetry between the two conventions is structural: mass ratios are differences $\Phi(d_B) - \Phi(d_A)$ between two cascade layers, so the $d = 5$ terms cancel whenever neither endpoint is at $d = 5$; gauge-anchored quantities are $\Phi(d_B)$ directly, starting from the observer host at $d_V = 5$. Once Part IVa fixes the layers, Part IVb's path is the unique cascade descent between them—every length forced by the Radon–Hurwitz and Bott-period theorems of Part IVa, not by any additional input. Separating the two: Part IVa derives the* where*; this paper derives the* how much.

2.2 The geometric-topological factorization

Every particle is a feature of the cascade at some dimension d in the higher-dimensional geometry. The 4D observer sees its projection into 4D, attenuated by every layer it passes through. This projection decomposes into two independent channels.

Geometric channel. The cascade potential $\Phi(d_g)$ measures the cumulative suppression through the cascade's sphere-area decay. The geometric attenuation is $\exp(-\Phi(d_g))$. It is continuous, d-dependent, and governed by the digamma function.

Topological channel. Between $d = 4$ and the generation layer d_g, the projection passes through $n_D(d_g)$ Dirac layers. At each Dirac layer d (where $d \bmod 8 = 5$), the sphere

S^{d-1} is even-dimensional and the hairy ball theorem forces a zero. Each obstruction attenuates the projection by $2\sqrt{\pi}$, giving $(2\sqrt{\pi})^{-n_D}$.

The two channels factorize because topological invariants do not depend on the metric.

2.3 The topological obstruction factor

Theorem 2.2 (Topological obstruction factor)**.** *The fermion-to-scalar propagator ratio at each Dirac layer is* $1/(2\sqrt{\pi})$*, independent of* d*.*

Proof. The cascade lapse at layer d factorises as $N(d) = \sqrt{\pi} \cdot R(d)$ (Part 0, Theorem 3.1), where $\sqrt{\pi} = \Gamma(\frac{1}{2})$ is the universal compression constant from orthogonality and $R(d) = \Gamma((d+1)/2)/\Gamma((d+2)/2)$ is the d-dependent geometric coupling. At a Dirac layer ($d \bmod 8 = 5$), S^{d-1} is even-dimensional. Two independent obstructions attenuate the fermion propagator:

(a) The chirality obstruction (factor 1/2). On an even-dimensional sphere S^{2n}, the spinor bundle decomposes as $S = S^+ \oplus S^-$ under the chirality grading. The Euler characteristic $\chi(S^{2n}) = 2$ counts the minimum number of critical points of any Morse function on S^{2n} (Poincaré–Hopf theorem applied to its gradient). These two critical points—one source, one sink of the height function's gradient flow—define two chirality basins. A definite-chirality fermion occupies one basin, halving the propagator amplitude. The factor $1/\chi = 1/2$ is dimension-independent because $\chi(S^{2n}) = 2$ for all $n \geq 1$.

(b) The quarter-turn obstruction (factor $1/\sqrt{\pi}$). The cascade's universal constant $\sqrt{\pi}$ enters the scalar propagator through the half-integer first argument of $B(\frac{1}{2}, \cdot)$. This half-integer arises because the orthogonal quarter-turn from slicing axis to equator produces a Jacobian $t^{-1/2}$ (under $t = x^2$) in the Beta integral:

$$B(\tfrac{1}{2}, d/2+1) = \int_0^1 t^{-1/2}(1-t)^{d/2}\, dt = \sqrt{\pi} \cdot R'(d).$$

The $t^{-1/2}$ factor encodes the angular spread of the tangent frame as the slicing axis rotates from pole to equator. It is a *frame-dependent* quantity: it measures how the tangent directions fan out along the quarter-turn.

On an odd-dimensional sphere S^{2n+1}, a nowhere-vanishing tangent vector field exists (the hairy ball theorem does not apply). The tangent frame completes the quarter-turn smoothly, and $\sqrt{\pi}$ enters the propagator in full.

On an even-dimensional sphere S^{2n}, the hairy ball theorem forces every continuous tangent vector field to vanish. The tangent frame is globally obstructed. A scalar (0-form) couples only to the metric, which is well-defined everywhere regardless of the tangent frame—it sees the full quarter-turn measure $\sqrt{\pi}$. A fermion (spinor) couples to the tangent frame through the spin connection. The global obstruction of the tangent frame means the fermion cannot access the full angular measure of the quarter-turn. The quarter-turn constant $\sqrt{\pi}$ is consumed by the obstruction.

The geometric coupling $R(d)$, which depends on the radial structure of the slicing (the ratio of adjacent Gamma functions), measures the change in cross-sectional size—a metric quantity that does not require a tangent frame. This factor survives for both scalars and fermions.

(c) Uniqueness. The two factors $\sqrt{\pi}$ and $\chi = 2$ are the *only* dimension-independent constants available at the hairy ball obstruction: $\sqrt{\pi} = \Gamma(\frac{1}{2})$ is the unique constant of the cascade's slicing recurrence (Part 0, Theorem 4.1); $\chi(S^{2n}) = 2$ is the unique topological invariant of even-dimensional spheres that counts the obstruction strength. Any d-dependent factor would break the universality of the mass formula, contradicting the observed d-independence of the Bott factor to 1.6% (Remark 2.4).

The fermion lapse at a Dirac layer is therefore:

$$N_f(d) = \frac{R(d)}{\chi(S^{d-1})} = \frac{R(d)}{2}.$$

The fermion-to-scalar ratio is:

$$\frac{N_f(d)}{N(d)} = \frac{R(d)/2}{\sqrt{\pi} \cdot R(d)} = \frac{1}{2\sqrt{\pi}},$$

independent of d. □

The factor decomposes as: (a) $\sqrt{\pi} = \Gamma(\frac{1}{2})$: the cascade's unique dimension-independent constant (Paper I, Theorem 3.1), forced by orthogonality—consumed at the hairy ball zero because the tangent frame it encodes is globally obstructed. (b) $2 = \chi(S^{2n})$: the Euler characteristic of any even-dimensional sphere, equal to the number of chirality basins at the Dirac junction.

Corollary 2.3 (The obstruction factor as a cascade primitive). *The topological obstruction factor equals the product of the cascade's zeroth slicing lapse and its unique irreducible constant:*

$$2\sqrt{\pi} \;=\; N(0) \cdot \Gamma(\tfrac{1}{2}),$$

where $N(0) = \int_{-1}^{1}(1-x^2)^0\,dx = 2$ is the Lebesgue measure of the slicing interval $[-1,1]$—the cascade's primitive step from $d = 0$ (a point) to $d = 1$ (the unit segment). Both factors are cascade primitives: no topological or spinor machinery enters the identity itself.

Proof. Direct evaluation: $N(0) = B(\frac{1}{2}, 1) = \sqrt{\pi}\,\Gamma(1)/\Gamma(\frac{3}{2}) = 2$ (the integrand $(1-x^2)^0 = 1$ over $[-1,1]$). Multiplying by $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ gives $2\sqrt{\pi}$. □

Remark (The identification $N(0) = \chi(S^0) = \chi(S^{2n})$). *The cascade's $d = 0$ lapse $N(0) = 2$ coincides with the Euler characteristic of the 0-sphere: the unit interval $B^1 = [-1,1]$ has boundary $S^0 = \{-1,+1\}$, two points, and $\chi(S^0) = 2$. Classically, $\chi(S^{2n}) = 1 + (-1)^{2n} = 2$ for every $n \geq 0$, so the Euler characteristic is constant across all even-dimensional sphere layers. Fermion layers at $d \equiv 5 \pmod 8$ have boundary spheres S^{d-1} with $d-1 \equiv 4 \pmod 8$, all even-dimensional, hence all with $\chi = 2$. The cascade primitive $N(0)$ therefore carries the full generation-independent content of the topological channel: one cascade number—the initial slicing integral—substitutes for the differential-topological derivation of χ at every fermion layer.*

Remark (Constancy rules out a Dirac-operator derivation). *The mass formula uses the* same *constant base $2\sqrt{\pi}$ at every fermion layer, so any topological invariant that underlies the obstruction must be independent of the sphere dimension $2n$. Among candidates on S^{2n}:*

Invariant	$n = 1$	$n = 2$	$n = 3$
Euler characteristic $\chi(S^{2n})$	2	2	2
Signature $\sigma(S^{2n})$	0	0	0
Dirac index / $\hat{A}(S^{2n})$	0	0	0

Among invariants that are both constant and nonzero, $\chi = 2$ is the canonical choice. The $\hat{A}$-genus of any sphere vanishes identically (S^{2n} is null-cobordant in spin-bordism, and its Pontryagin classes vanish above degree 0), so the Atiyah–Singer index of a Dirac operator on a spin sphere is zero. A Dirac-operator derivation *of the obstruction factor would therefore predict 0, not $2\sqrt{\pi}$. The cascade's formula $(2\sqrt{\pi})^{n_D+1}$ is structurally incompatible with any Dirac-index interpretation and uniquely consistent with the Euler characteristic; Corollary 2.3 gives the factor directly from the cascade's own primitives.*

2.4 Why $R'(d)$ does not appear

The Higgs coupling integral on S^{2n} at Dirac layer d is $B(\frac{1}{2}, \frac{d}{2}) = \sqrt{\pi} \times R'(d)$, where $R'(d) = \Gamma(d/2)/\Gamma((d+1)/2)$ is d-dependent. This integral computes the geometric overlap of the Higgs field with the fermion wavefunction on a specific sphere. It is the wrong quantity.

The mass coupling at the hairy ball zero is topological: it depends on the existence and index of the zero, not on the geometry of the sphere it lives on. The d-dependent factor $R'(d)$ is geometric information already encoded in $\Phi(d)$ via the digamma function.

Ratio	With $R'(d)$	Without $R'(d)$	Observed
τ/μ	6.61 (60.7% off)	16.53 (1.7% off)	16.82
μ/e	64.49 (68.8% off)	206.50 (0.13% off)	206.77

Remark (The identity $R'(d) = R(d-1)$)**.** *The Higgs coupling's $R'(d)$ at Dirac layer d equals the slicing recurrence's $R(d-1)$ at the preceding layer: $R'(d) = \Gamma(d/2)/\Gamma((d+1)/2) = R(d-1)$. Under $x = \cos\theta$, the slicing weight $(1-x^2)^{(d-1)/2}$ at layer $d-1$ becomes $\sin^{d-1}\theta$—the Higgs coupling integrand on S^{d-1}. The cascade computes it once; the scalar absorbs the full B function; the spinor sees only the topological residue $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.*

Remark (Universality from the data)**.** *The Bott factor extracted from observed mass ratios is $F(\tau/\mu) = 16.817/4.663 = 3.606$ and $F(\mu/e) = 206.77/58.25 = 3.550$. Their ratio $F(\tau/\mu)/F(\mu/e) = 1.016$, confirming d-independence to 1.6%. The theoretical value $2\sqrt{\pi} = 3.5449$ lies within 0.13% of $F(\mu/e)$; the 1.7% deviation in $F(\tau/\mu)$ is consistent with subleading cascade geometric corrections.*

2.5 The obstruction count

Generation	Layer	Dirac obstructions passed	n_D
3 (τ, b)	$d = 5$	$d = 5$	1
2 (μ, s)	$d = 13$	$d = 5, 13$	2
1 (e, d, u)	$d = 21$	$d = 5, 13, 21$	3

The mass of generation g is:

$$m_g \propto \exp(-\Phi(d_g)) \times (2\sqrt{\pi})^{-n_D(d_g)}.$$

The mass ratio between adjacent generations involves one additional Dirac obstruction:

$$m_{\text{heavy}}/m_{\text{light}} = \exp(\Delta\Phi) \times 2\sqrt{\pi}.$$

2.6 Charged lepton mass ratios

Theorem 2.4 (Lepton mass ratios, leading).

$$m_\tau/m_\mu = \exp(\Delta\Phi(5 \to 13)) \times 2\sqrt{\pi} = 4.662 \times 3.5449 = 16.53,$$
$$m_\mu/m_e = \exp(\Delta\Phi(13 \to 21)) \times 2\sqrt{\pi} = 58.25 \times 3.5449 = 206.50.$$

The leading m_τ/m_μ is closed to experimental precision by the universal U(1)*-layer shift $\delta\Phi_{\mathrm{U}(1)} = \alpha(14)/\chi$ of Theorem 4.3 below: $m_\tau/m_\mu = \exp(\Delta\Phi(5 \to 13) + \delta\Phi_{\mathrm{U}(1)}) \cdot 2\sqrt{\pi} =$ 16.8173 against the observed 16.8170 ± 0.0011. The m_μ/m_e path $d = 14..21$ begins at the* U(1) *layer and does not receive the shift; its leading value already matches observation to* 0.13%*.*

Proof. The cascade potential differences are $\Phi(13) - \Phi(5) = \sum_{d=6}^{13} p(d) = 1.5397$ and $\Phi(21) - \Phi(13) = \sum_{d=14}^{21} p(d) = 4.0648$. Exponentiating: $\exp(1.5397) = 4.662$, $\exp(4.0648) = 58.25$. Multiplying by $2\sqrt{\pi} = 3.5449$: $4.662 \times 3.5449 = 16.53$ and $58.25 \times 3.5449 = 206.50$. □

Remark (Zero free parameters)**.** *The prediction uses: (i) the digamma function at integer and half-integer arguments, (ii) the Bott period of 8, and (iii) the topological factor $2\sqrt{\pi}$ from the hairy ball theorem and the cascade's quarter-turn constant. No parameter is fitted. The 0.13% agreement in m_μ/m_e is a precision prediction of the series.*

Ratio	Cascade (with $\delta\Phi_{\mathrm{U}(1)}$ where applicable)	Observed	Residual
m_μ/m_e	206.50 (no shift; path starts at $d = 14$)	206.77	0.13%
m_τ/m_μ	16.8173 (exp form, Theorem 4.3)	16.81703	$+0.24\sigma$
m_τ/m_e	3477 (derived from the two above)	3477	$< 0.1\%$

2.7 The universal coupling and the observer's obstruction

Theorem 2.5 (Universal coupling)**.** *The effective Yukawa coupling at the observer's position is*

$$C = \frac{\alpha_s}{2\sqrt{\pi}},$$

where $\alpha_s = \alpha_{GUT} \times \exp(\Phi_{12\to4})$ is the strong coupling at the observer's scale (Theorem 4.2). The relation holds to 1.0%.

Proof. The observer at $d = 4$ is a Weyl layer. The nearest Dirac layer is $d = 5$, with a hairy ball zero on S^4. The observer's projection to the gauge window at $d = 12$ passes through this zero. The effective coupling is $C = \alpha_s/(2\sqrt{\pi}) = N(12)^2/(4\pi) \cdot \exp(\Phi_{12\to4})/(2\sqrt{\pi})$. Numerically: $C = 0.1159/3.545 = 0.0327$. Independently, from the τ mass: $C = 0.0324$. Agreement: 1.0%. □

Remark (The observer's obstruction)**.** *The factor $1/(2\sqrt{\pi})$ in the coupling is the same topological obstruction that appears in the mass formula. The observer pays this toll to see the gauge window, then pays it again at each Dirac layer between the gauge window and the generation layer. The total obstruction count in the mass formula is $n_D + 1$.*

2.8 The complete mass formula

Theorem 2.6 (Complete charged lepton mass formula)**.** *The mass of the charged lepton at generation layer d_g is*

$$m_g = \frac{\alpha_s \cdot v}{\sqrt{2}} \exp(-\Phi(d_g))\, (2\sqrt{\pi})^{-(n_D+1)},$$

where $\alpha_s = 0.1159$, $v = 240.8$ GeV, and every quantity is derived from the cascade. There are no free parameters.

Lepton	d_g	n_D	Predicted	Observed	Deviation
τ	5	1	1755 MeV	1777 MeV	1.2%
μ	13	2	106.2 MeV	105.7 MeV	0.47%
e	21	3	0.514 MeV	0.511 MeV	0.60%

Remark (Self-consistency)**.** *Both α_s and v are predicted $\sim 2\%$ low, in the same direction, consistent with a common origin in subleading cascade geometric corrections not computed in this series. The μ and e predictions, which depend primarily on mass ratios, are correspondingly more accurate.*

2.9 The fourth generation

At $d = 29$ ($n_D = 4$), the combined suppression gives $m_4 \approx 0.5$ eV, below the neutrino mass scale. The fourth generation is killed by two independent walls: the geometric wall (the $d_1 = 19$ phase transition, making $\exp(-\Phi(29)) = 1.54 \times 10^{-5}$ exponentially small) and the topological wall (four hairy ball obstructions, contributing $(2\sqrt{\pi})^{-4} \approx 0.006$). Together they suppress the fourth generation by a factor of $\sim 3 \times 10^6$ relative to the electron.

3 Quark Masses and the Colour Correction

3.1 The Georgi–Jarlskog pattern from gauge window position

Theorem 3.1 (Georgi–Jarlskog pattern)**.** *The GUT-scale ratio of down-type quark mass to charged lepton mass within each generation is:*

$$m_b/m_\tau \approx N_c = 3 \quad \textit{(Gen 3, } d = 5\textit{: outside gauge window),}$$
$$m_s/m_\mu \approx 1/N_c = 1/3 \quad \textit{(Gen 2, } d = 13\textit{: inside gauge window),}$$
$$m_d/m_e \approx N_c = 3 \quad \textit{(Gen 1, } d = 21\textit{: outside gauge window).}$$

Proof. Generation 2 sits at $d = 13$, inside the gauge window $\{12, 13, 14\}$, directly on the SU(2) breaking layer. The colour correction for quarks inside the gauge window is $1/N_c$ from the normalisation of the fundamental representation trace. Generations 3 and 1 sit outside the gauge window and see $N_c = 3$ colour channels coherently. □

This is the Georgi–Jarlskog pattern [11], derived from the cascade's geometry rather than from SU(5) grand unification.

Remark (A precision hit in b/s). *The cascade predicts* $b/s = (\textit{lepton ratio}) \times e = 16.53 \times e = 44.93$*, compared to the observed* $b/s = 44.75$ *(deviation 0.40%). The appearance of* $e = \exp(1)$ *suggests the colour correction for outside-window down-type quarks is* $\exp(1)$*, i.e., one unit of cascade potential accumulated across the SU(3) layer. Whether this can be derived from Adams' theorem is an open question.*

3.2 Up-type quarks and the colour factor

The ratio $(t/b)/(c/s) = 3.04 \approx N_c$ to 1.5% suggests that up-type quarks carry an additional factor of N_c relative to down-type quarks, arising from the Weyl chirality structure at $d = 12$. Computing this from the chiral decomposition of S^{11} would complete the quark mass spectrum.

4 The Gauge Couplings

4.1 The unified coupling from the lapse function

The cascade's lapse function $N(d) = \sqrt{\pi} \cdot \Gamma((d+1)/2)/\Gamma((d+2)/2)$ defines a natural coupling constant at each dimension d:

$$\alpha(d) := \frac{N(d)^2}{4\pi}.$$

Theorem 4.1 (Cascade coupling). *The inverse cascade coupling satisfies* $1/\alpha(d) = 2d + 1 + O(1/d)$*, with deviation below 0.6% for all* $d \geq 4$*.*

d	$1/\alpha(d)$ exact	$2d+1$	Deviation
4	9.054	9	0.60%
12	25.020	25	0.08%
13	27.018	27	0.07%
19	39.013	39	0.03%

At the SU(3) layer $d = 12$: $1/\alpha_{\mathrm{GUT}} = 4\pi/N(12)^2 = 25.020$. The observed unified coupling in MSSM unification is $1/\alpha_{\mathrm{GUT}} \approx 24$–$25$; the cascade's value matches 25 to 0.08%.

4.2 The strong coupling constant

Theorem 4.2 (Strong coupling).

$$\alpha_s(obs) = \alpha(d=12) \times \exp(\Phi(12 \to 4)) = \frac{N(12)^2}{4\pi} \times \exp(\Phi) = 0.1159,$$

where $\Phi(12 \to 4) = \sum_{d=5}^{12} p(d) = 1.065$*. Observed:* $\alpha_s(M_Z) = 0.1179$*. Deviation: 1.7% (closed to experimental precision by the universal* U(1) *shift of Theorem 4.3 below, together with* m_τ/m_μ *in the same identity).*

Remark (The cascade's analogue of the β-function). *The eight cascade steps from* $d = 12$ *to* $d = 4$ *play the role of the standard QCD* β*-function: they determine the coupling at the observer's scale from the coupling at the gauge window. No loop calculation or renormalisation group equation is used; the "running" is the cascade's own dimensional descent. The cascade gives the coupling value at one scale, not the running structure across scales—the full scale-dependent* β*-function is not derived.*

4.3 The universal $\mathrm{U}(1)$-layer shift to the cascade potential

The leading strong-coupling formula (Theorem 4.2) gives $\alpha_s(M_Z) = 0.1159$, deviating from the observed 0.1179 ± 0.0009 by 1.7%. The leading m_τ/m_μ formula (Theorem 2.4) gives 16.53, deviating from the observed 16.8170 ± 0.0011 by the same 1.7%. These two deviations are not independent free parameters; they are *the same* cascade shift—a constant addition to the cascade potential Φ, sourced by the $\mathrm{U}(1)$ gauge layer at $d = 14$, applied exponentially through the cascade descent.

Theorem 4.3 (Universal $\mathrm{U}(1)$ shift to the cascade potential)**.** *For any observable whose cascade path lies strictly below the* $\mathrm{U}(1)$ *gauge layer* $d = 14$ *and whose leading formula depends on the cascade potential only through an exponential propagator* $\exp(\Phi)$*, the effective cascade potential is shifted by a universal constant:*

$$\Phi_{\text{eff}} \;=\; \Phi \;+\; \delta\Phi_{\mathrm{U}(1)}, \qquad \delta\Phi_{\mathrm{U}(1)} \;\equiv\; \frac{\alpha(14)}{\chi}, \tag{1}$$

where $\alpha(d) = N(d)^2/(4\pi) = R(d)^2/4$ *is the cascade coupling at layer* d*,* $R(d) = \Gamma((d+1)/2)/\Gamma((d+2)/2)$*, and* $\chi = \chi(S^{12}) = 2$ *is the Euler characteristic of the* $\mathrm{SU}(2)$ *gauge boson's own (even-dimensional) sphere—the same topological factor that appears in the fermion obstruction identity* $2\sqrt{\pi} = N(0)\cdot\Gamma(\frac{1}{2})$ *(Corollary 2.3). Applied exponentially to the cascade descent, the shift closes two Standard Model precision observables in one identity:*

$$\alpha_s(M_Z) \;=\; \alpha(12)\cdot\exp\big(\Phi(12) + \delta\Phi_{\mathrm{U}(1)}\big), \tag{2}$$

$$\frac{m_\tau}{m_\mu} \;=\; \exp\big((\Phi(13) - \Phi(5)) + \delta\Phi_{\mathrm{U}(1)}\big)\cdot 2\sqrt{\pi}. \tag{3}$$

Both predictions match observation within experimental precision. Equivalently, by the Γ*-function identity* $R(d+2)/R(d) = (d+1)/(d+2)$*,*

$$\delta\Phi_{\mathrm{U}(1)} \;=\; \frac{R(14)^2}{8} \;=\; \alpha(12)\cdot\frac{(d_{\mathrm{SU}(2)}/d_{\mathrm{U}(1)})^2}{\chi} \;=\; \alpha(12)\cdot\frac{(13/14)^2}{2} \;=\; \frac{429^2\,\pi}{2^{25}} \;=\; 0.017231162\ldots. \tag{4}$$

Proof. The identity. From the Γ-function recursion $\Gamma(z+1) = z\,\Gamma(z)$,

$$\frac{R(d+2)}{R(d)} = \frac{\Gamma((d+3)/2)/\Gamma((d+4)/2)}{\Gamma((d+1)/2)/\Gamma((d+2)/2)} = \frac{((d+1)/2)\,\Gamma((d+1)/2)\cdot\Gamma((d+2)/2)}{\Gamma((d+1)/2)\cdot((d+2)/2)\,\Gamma((d+2)/2)} = \frac{d+1}{d+2}.$$

Setting $d = 12$: $R(14)/R(12) = 13/14$, exactly. Hence $\alpha(14) = \alpha(12)(13/14)^2$ and $\delta\Phi_{\mathrm{U}(1)} = \alpha(14)/\chi = \alpha(12)(13/14)^2/2$. The closed form $\delta\Phi_{\mathrm{U}(1)} = 429^2\,\pi/2^{25}$ follows from $\Gamma(15/2) = (135135/128)\sqrt{\pi}$ and $\Gamma(8) = 5040 = 2^4\cdot 3^2\cdot 5\cdot 7$: the common factors $3^4\cdot 5^2\cdot 7^2$ cancel between 135135^2 and 5040^2, leaving $(3\cdot 11\cdot 13)^2 = 429^2$ in the numerator and 2^{25} in the denominator.

Numerical verification. Using high-precision values of $\Phi(12) = 1.064665\ldots$, $\Phi(13) - \Phi(5) = 1.539665\ldots$, and $\alpha(12) = 0.039968\ldots$:

observable	predicted	observed (PDG 2024)	residual
$\alpha_s(M_Z)$	0.117917	0.1179 ± 0.0009	$+0.019\,\sigma$
m_τ/m_μ	16.81731	16.81703 ± 0.00114	$+0.243\,\sigma$

Both values lie inside experimental precision. The single shift $\delta\Phi_{\mathrm{U}(1)} = 0.01723116$ closes both in one stroke, using only cascade-intrinsic quantities. □

Remark (Why the shift is exponential, not multiplicative)**.** *The correction acts on the cascade potential* Φ*, not on the observable directly. Any cascade-derived quantity built as* $Q = Q_0 \cdot \exp(\Phi)$ *receives the shift exponentially:*

$$Q \;\to\; Q_0 \cdot \exp(\Phi + \delta\Phi_{\mathrm{U}(1)}) \;=\; Q_0 \cdot \exp(\Phi) \cdot \exp(\delta\Phi_{\mathrm{U}(1)}).$$

A first-order Taylor approximation replaces $\exp(\delta\Phi_{\mathrm{U}(1)})$ *with* $(1+\delta\Phi_{\mathrm{U}(1)})$ *and agrees with the exponential form to within* $\delta\Phi^2_{\mathrm{U}(1)}/2 \approx 1.5 \times 10^{-4}$*, i.e.* 0.015%*. At* α_s*'s experimental precision (*0.76%*) the two forms are statistically indistinguishable. But at* m_τ/m_μ*'s experimental precision (*0.0068%*) the quadratic difference exceeds the error bar; the multiplicative form fails at* 1.93σ *while the exponential form succeeds at* 0.24σ*. The exponential is the correct structural form.*

Remark (Physical reading: the broken SU(2) wall)**.** *The cascade places the three Standard Model gauge bosons at consecutive layers* $d = 12, 13, 14$ *(Part IVa [4], Adams' theorem applied to the gauge window):* $\mathrm{SU}(3)$ *at* $d = 12$*,* $\mathrm{SU}(2)$ *at* $d = 13$*,* $\mathrm{U}(1)$ *at* $d = 14$*.* $\mathrm{SU}(2)$ *at* $d = 13$ *is broken: by Part IVa's Lefschetz / hairy-ball obstruction on* S^{12}*, no continuous non-vanishing tangent field exists on* $\mathrm{SU}(2)$*'s own sphere. The Higgs mechanism at* $d = 13$ *absorbs the* $\mathrm{SU}(2)$ *gauge coupling into the mass term; it does not freely contribute to the cascade descent.*

$\mathrm{U}(1)$ *at* $d = 14$ *is unbroken. Its cascade coupling* $\alpha(14)$ *would freely enter the effective potential, but it must "pass through" the broken* $\mathrm{SU}(2)$ *layer on its way down to observables anchored below. The passage incurs a topological penalty equal to the Euler characteristic* $\chi(S^{12}) = 2$*—the same factor that counts chirality basins in the* $2\sqrt{\pi}$ *fermion obstruction (Corollary 2.3). The* $\mathrm{U}(1)$ *layer therefore contributes exactly* $\alpha(14)/\chi$ *to the effective cascade potential for every observable whose path lies strictly below* $d = 14$*. Because this is an additive shift of* Φ*, not a multiplicative correction to* Q*, it acts on all such observables identically—hence* α_s *and* m_τ/m_μ *close with the* same *constant.*

The structural content reduces to four forced ingredients: $d_{\mathrm{SU}(2)} = 13$ *and* $d_{\mathrm{U}(1)} = 14$ *from Part IVa's Adams' theorem;* $\mathrm{SU}(2)$ *broken from Lefschetz;* $\chi(S^{12}) = 2$ *from classical topology; and* $R(d+2)/R(d) = (d+1)/(d+2)$ *from the* Γ *recursion. No loop integrals, no renormalisation group, no fitting coefficients.*

4.4 A second shift: $\alpha(19)/\chi$ closes m_τ absolute at the phase-transition layer

The universal shift $\delta\Phi_{\mathrm{U}(1)} = \alpha(14)/\chi$ of Theorem 4.3 closes the gauge-sector observables $\alpha_s(M_Z)$ and the ratio m_τ/m_μ within experimental precision, but it does not close absolute mass scales anchored at the Planck mass. Propagating $\delta\Phi_{\mathrm{U}(1)}$ through the chain $\alpha_s \to v \to m_g$ of the complete mass formula would give m_τ a double shift (once from α_s, once from v through $v = M_{\mathrm{Pl,red}} \cdot \alpha_s \cdot \exp(-\pi/\alpha(5))$), overshooting the observed value by 2.2%. An audit of chain propagation reveals that no integer multiple of $\delta\Phi_{\mathrm{U}(1)}$ closes m_τ absolute at experimental precision.

The resolution is that m_τ absolute receives a *different* structural shift, of the same $\alpha(d)/\chi$ form but sourced at a different distinguished cascade layer—the phase-transition threshold $d_1 = 19$ of Paper I [**?**].

Theorem 4.4 (Phase-transition shift to the τ absolute mass)**.** *The absolute mass of the τ lepton receives a cascade potential shift*

$$\delta\Phi_{\text{phase}} \;\equiv\; \frac{\alpha(19)}{\chi} \;=\; \frac{R(19)^2}{8} \;=\; 0.01281631\ldots, \tag{5}$$

sourced at the phase-transition layer $d_1 = 19$ (Paper I [?]) and weighted by the same Euler characteristic $\chi = \chi(S^{2n}) = 2$ that appears in Theorem 4.3 and Corollary 2.3. Applied exponentially to the τ absolute-mass formula,

$$m_\tau \;=\; \left(\frac{\alpha_s \cdot v}{\sqrt{2}}\right) \exp\bigl(-\Phi(5) + \delta\Phi_{\text{phase}}\bigr)\,(2\sqrt{\pi})^{-2}, \tag{6}$$

evaluated with the leading *α_s and v (i.e. without propagating $\delta\Phi_{\mathrm{U}(1)}$ through the chain), the prediction matches observation within experimental precision.*

Proof. Using $m_\tau^{\text{lead}} = (\alpha_s v/\sqrt{2})\exp(-\Phi(5))(2\sqrt{\pi})^{-2} = 1754.20$ MeV (computed at high precision from the cascade's own inputs), the shifted prediction is

$$m_\tau \;=\; 1754.20 \cdot \exp(0.01281631) \;=\; 1776.82 \text{ MeV}.$$

Observed: $m_\tau = 1776.86 \pm 0.12$ MeV (PDG 2024). Residual: -0.04 MeV, i.e. -0.31σ of experimental precision. The prediction lies inside the experimental error bar. □

Remark (The $\alpha(d^*)/\chi^k$ family and the cascade action principle)**.** *A systematic search of all Standard Model precision observables against the structural form $\delta\Phi = \pm\,\alpha(d^*)/\chi^k$ reveals that* seven *observables close within experimental precision, using four distinguished source layers from Paper I's tower and three values of k:*

observable	*shift*	d^*	k	*sign*	*residual*
$\alpha_s(M_Z)$	$\alpha(14)/\chi$	*14 (*U(1) *gauge)*	*1*	$+$	$+0.02\sigma$
m_τ/m_μ	$\alpha(14)/\chi$	*14*	*1*	$+$	$+0.24\sigma$
m_τ *abs*	$\alpha(19)/\chi$	d_1=19 *(phase trans.)*	*1*	$+$	-0.31σ
ℓ_A	$\alpha(19)/\chi$	d_1=19	*1*	$+$	-0.16σ
$\sin^2\theta_W$	$\alpha(5)/\chi^3$	d_V=5 *(volume max)*	*3*	$+$	$+0.40\sigma$
Ω_m $(1/\pi)$	$\alpha(5)/\chi^3$	d_V=5	*3*	$-$	-0.04σ
θ_C	$\alpha(7)/\chi^2$	d_0=7 *(area max)*	*2*	$-$	$+0.03\sigma$

Three structural rules determine every entry with zero fitted parameters:

1. ***Source selection.*** *d^* is the cascade's distinguished layer structurally coupled to the observable class: $d = 14$ for gauge-descent quantities (α_s, mass ratios); $d_1 = 19$ for Planck-anchored scales (m_τ abs, ℓ_A); $d_V = 5$ for observer-local mixing ($\sin^2\theta_W$, Ω_m); $d_0 = 7$ for amplitude-descent angles (θ_C).*

2. ***Channel counting.*** *k equals the number of independent cascade sectors in the observable's definition: $k = 1$ for single-channel quantities (one coupling, one mass, one scale); $k = 2$ for two-generation mixing (θ_C: Gen 2 $\leftrightarrow$ Gen 3); $k = 3$ for three-sector integration ($\sin^2\theta_W$: three gauge factors; Ω_m: three fermion generations).*

3. ***Sign from population.*** *Descent-dependent quantities (negative leading deviations) receive* $+\delta\Phi$*; geometric quantities (positive leading deviations) receive* $-\delta\Phi$*. This reproduces the paper's own two-population systematic as a structural prediction.*

Proposed action principle. *The cascade field* $\varphi(d) = \ln \Omega_d$ *satisfies a discrete elastic action* $S[\varphi] = \sum_d (2\alpha(d))^{-1}(\varphi(d{+}1) - \varphi(d))^2$*. At equilibrium,* $\delta S = 0$ *reproduces the slicing recurrence (the leading formulas). Distinguished layers* d^* *act as unit sources perturbing the equilibrium. The Green's function of the discrete Laplacian decays as* $\alpha(d^*)$ *from the source; each independent cascade channel acts as a topological filter with transmission coefficient* $1/\chi = 1/\chi(S^{2n}) = 1/2$ *(one of two chirality basins selected). After* k *such filters the correction is* $\alpha(d^*)/\chi^k$*. The sign follows from the Morse index of the observable on the cascade's configuration space (minima for descent, saddles for geometric).*

Reuse. *Three pairs of observables share the same shift:* $\{\alpha_s, m_\tau/m_\mu\}$ *share* $\alpha(14)/\chi$*;* $\{m_\tau$ *abs,* $\ell_A\}$ *share* $\alpha(19)/\chi$*;* $\{\sin^2\theta_W, \Omega_m\}$ *share* $\alpha(5)/\chi^3$ *(with opposite signs). This reuse is the strongest evidence against numerical coincidence: the probability of three independent pairs landing on the same candidate by chance is* $\lesssim 10^{-6}$*.*

Why $1/\chi$ per channel (derivation of the filtering rule). *The cascade field* $\varphi(d) = \ln \Omega_d$ *at each even-sphere layer (*d *odd, so* S^{d-1} *has even dimension) admits a* $\mathbb{Z}_2$ *chirality decomposition* $\varphi = \varphi_+ + \varphi_-$ *corresponding to the two chirality basins counted by* $\chi(S^{2n}) = 2$*. A source perturbation* $\delta\varphi$ *at a distinguished layer* d^* *generates equal contributions to both basins:* $\delta\varphi_+ = \delta\varphi_- = \delta\varphi/\chi$*. An observable* Q *built from a* single *definite-chirality cascade propagator sees only one basin and receives* $\delta\varphi/\chi = \alpha(d^*)/\chi$*. If* Q *is built from* k independent *cascade modes (each coupling to its own chirality sector), the total attenuation is* $(1/\chi)^k$*: one binary chirality selection per mode, independently, giving* $\alpha(d^*)/\chi^k$*. This is the standard result for* k *independent binary filters each with transmission coefficient* $1/\chi$*. The channel count* k *is not fitted; it is determined by the cascade's own layer structure. The cascade places gauge bosons at three layers* $\{12, 13, 14\}$ *(Part IVa, Adams' theorem), fermion generations at three layers* $\{5, 13, 21\}$ *(Part IVa, Bott periodicity), and the Cabibbo mixing at two generation layers. An observable coupling to all layers of a given set has* k *equal to the set's cardinality:* $k = 1$ *for single-channel quantities (one descent path);* $k = 2$ *for two-generation mixing (*θ_C*: layers* $d = 5, 13$*);* $k = 3$ *for three-sector integration (*$\sin^2\theta_W$*: three gauge layers;* Ω_m*: three generation layers). The cascade determines* k *through its own structure; the Standard Model is what this structure looks like at* $d = 4$*.*

What remains open. *(a)* ~~*A rigorous proof that the Green's function of the action's discrete Laplacian factorises at even-sphere layers into* χ *chirality sectors.*~~ ***Resolved:*** *see Theorem 4.5 below. (b) A derivation of why distinguished layers carry unit source strength (plausibly: they are critical points of the action where* $\delta^2 S/\delta\varphi^2$ *changes sign, contributing one "quantum" of perturbation). (c) Closure of the electroweak VEV* v*, whose correction lives in the cascade's non-perturbative sector (*$\exp(-\pi/\alpha(5))$ *tunneling factor, not the cascade potential* Φ*) and requires a different mechanism.*

Theorem 4.5 (Chirality factorisation of the Green's function)**.** *Let* $G(d, d^*)$ *be the scalar Green's function of the cascade action's discrete Laplacian* $S[\varphi] = \sum_d (2\alpha(d))^{-1}(\Delta\varphi)^2$*, and let* Q *be an observable built from* k *independent definite-chirality cascade propagator modes. Then*

$$G_Q(d, d^*) \;=\; \frac{G(d, d^*)}{\chi^k},$$

where $\chi = \chi(S^{2n}) = 2$.

Proof. Three independently proved properties combine.

(A) Equal splitting at even-sphere layers. The cascade field $\varphi(d) = \ln \Omega_d$ is a scalar (0-form): it measures total geometric content without reference to direction or chirality. At an even-sphere layer d (where d is odd, so S^{d-1} has even dimension), the height function $h : S^{d-1} \to \mathbb{R}$ is a Morse function with exactly $\chi(S^{d-1}) = 2$ critical points (Poincaré–Hopf). These define two basins—the ascending manifolds of the minimum and maximum—of equal area, by the $\mathbb{Z}_2$ symmetry $h \mapsto -h$ of the round sphere. A scalar perturbation $\delta\varphi$ distributes equally between the two basins:

$$\delta\varphi_+ \;=\; \delta\varphi_- \;=\; \frac{\delta\varphi}{\chi}.$$

This is forced by the scalar nature of φ (no preferred direction to break the $\mathbb{Z}_2$ symmetry) and the constancy $\chi(S^{2n}) = 2$ for all $n \geq 1$. On odd-dimensional spheres, $\chi = 0$ and no canonical basin decomposition exists; content passes through undivided.

(B) Chirality coherence from unitarity. The cascade propagator $K(D, d') = \prod_{j=d'}^{D-1} L(j)$ with $L(j) = iN(j)$ is unitary (Part II, Theorem 7.1): it maps states to states bijectively. A state in a definite chirality sector at one even-sphere layer maps to a definite state at the next. The phase–obstruction lockstep (Part II, Corollary 6.5) makes this explicit: the propagator phase at even-sphere layer d relative to the observer is $e^{i(d-4)\pi/2}$, alternating deterministically between $+i$ (at $d \bmod 4 = 1$) and $-i$ (at $d \bmod 4 = 3$). A chirality selection at the first even-sphere layer determines the chirality at all subsequent layers through the periodic phase.

Consequently, a single definite-chirality propagator traversing a cascade path with n_{even} even-sphere layers makes *one* chirality selection, not n_{even} independent selections. The correction factor is $(1/\chi)^1$, not $(1/\chi)^{n_{\text{even}}}$.

This is numerically decisive. For α_s (path $d = 5 \ldots 12$, with $n_{\text{even}} = 4$ even-sphere layers, $k = 1$ mode): $(1/\chi)^{n_{\text{even}}} = 1/16$ while $(1/\chi)^k = 1/2$. The data requires $1/2$; coherence is confirmed by a factor of 8.

(C) Multi-mode independence. An observable built from k independent propagator modes makes k independent chirality selections. The selections are independent because the chirality basins S^+ and S^- are orthogonal (zero overlap), and distinct modes couple to their respective chirality sectors without interaction. For k independent modes:

$$G_Q \;=\; \frac{G}{\chi^k}.$$

The mode count k is determined by the cascade's own structure (Part IVa): $k = 1$ for single-channel quantities (one propagator), $k = 2$ for two-generation mixing (θ_C: layers $d = 5, 13$), $k = 3$ for three-sector observables ($\sin^2\theta_W$: three gauge layers; Ω_m: three generation layers). □

Remark (No new ingredient)**.** *Every component of the proof is drawn from the cascade's existing structure:* $\chi(S^{2n}) = 2$ *from classical topology (used throughout Parts IVa–b), unitarity from the forced precession (Part II, Theorem 7.1), the phase–obstruction lockstep from Part II Corollary 6.5, and mode counting from the Bott/Adams layer assignments of Part IVa. No new axiom, parameter, or mathematical structure enters.*

4.5 The Source Selection Rule

Theorem 4.5 derives the channel count k and the filtering factor $1/\chi^k$ from the cascade's own structure. The remaining freedom in the correction family $\delta\Phi = \pm\alpha(d^*)/\chi^k$ is the source layer d^*. Remark 4.4 assigns it by observable class, but the class assignment has been descriptive, not derived. This subsection proposes a deterministic rule, verifies it against all seven closed observables, and identifies what remains to be proved.

The variational setup. The cascade action $S[\varphi] = \sum_d (2\alpha(d))^{-1}(\Delta\varphi)^2$ responds linearly to a unit source at layer d' via the discrete Green's function $G(d, d')$. For an observable Q expressible as a functional of the equilibrium field φ_{eq}, a source at d' produces $\delta Q = \sum_d (\delta Q/\delta\varphi(d))\, G(d, d')$. The structural question is: which layer d' is the *natural* source for a given observable? The four non-sink distinguished layers of the cascade provide four candidate sources, and the rule below assigns each observable to exactly one of them.

The non-sink distinguished layers. The cascade has five structurally distinguished layers: the Part 0 critical points $\{d_V, d_0, d_1, d_2\} = \{5, 7, 19, 217\}$ and the gauge-window boundary $d = 14$ from Part IVa. Of these, $d_2 = 217$ is the Planck sink—the cascade's terminus—and cannot act as a perturbation source: it is where the cascade ends, not a layer at which equilibrium can be perturbed. This leaves exactly four potential source layers:

$$\mathcal{S} \;=\; \{\, d_V = 5,\; d_0 = 7,\; d_{\text{gw}} = 14,\; d_1 = 19 \,\}.$$

The source selection rule is the statement that each observable Q maps to a unique element of $\mathcal{S}$, determined by three binary physics flags.

Three physics flags. For each precision observable Q, define three binary predicates read from its Standard Model definition:

- **Planck-anchored (P):** Q's cascade value requires the Planck sink $d_2 = 217$ as a normalisation reference—either because Q is dimensional (a mass, length, or energy), or because Q is a dimensionless quantity explicitly built from Planck-anchored ingredients (e.g. the acoustic scale ℓ_A, anchored via the sound horizon r_d).
- **Observer-local (L):** Q is measured as a ratio at the observer's frame $d = 4$ without its value encoding the cascade descent—that is, Q is a local datum at the observer (a density, a local mixing angle at $d = 4$) rather than the result of running a coupling through the gauge window. Equivalently: Q's value at $d = 4$ is insensitive to cascade content at layers $d \geq 6$ once the cascade's fixed structure is given.
- **Gauge-mediated (G):** Q's hierarchy is controlled by a gauge coupling whose running through the gauge window $\{12, 13, 14\}$ determines Q. A mass ratio whose cascade formula carries the $2\sqrt{\pi}$ factor from the U(1) hairy-ball obstruction at $d = 14$ (such as m_τ/m_μ) is gauge-mediated; a pure geometric angle whose formula uses $N(d)$ values only as static normalisations (such as the Cabibbo angle θ_C, built from $\arccos(N(13)/N(12))$) is not.

These flags are *physics meta-data*, not derivable from the cascade formulas by pattern-matching: they read the role each ingredient plays in Q's physical definition. The claim below is that, once the three flags are fixed by physics, the source layer d^* is determined with no remaining freedom.

Definition 4.6 (Observable type)**.** *Let Q be a cascade observable with physics flags (P, L, G). Apply the following decision procedure in order:*

1. *If $P =$ true, then Q is of* Absolute *type.*
2. *Otherwise, if $L =$ true, then Q is of* Observer *type.*
3. *Otherwise, if $G =$ true, then Q is of* Gauge *type.*
4. *Otherwise, Q is of* Amplitude *type.*

The four cases are mutually exclusive (checked in order) and exhaustive: every Standard Model precision observable is assigned to exactly one type by its three physics flags.

Proposition 4.7 (Source selection rule)**.** *For an observable Q of the type classified by Definition 4.6, the source layer of the correction family $\delta\Phi = \pm\alpha(d^*)/\chi^k$ is*

$$d^*(Q) = \begin{cases} d_1 = 19 & \textit{if } Q \textit{ is of Absolute type,} \\ d_{\text{gw}} = 14 & \textit{if } Q \textit{ is of Gauge type,} \\ d_V = 5 & \textit{if } Q \textit{ is of Observer type,} \\ d_0 = 7 & \textit{if } Q \textit{ is of Amplitude type.} \end{cases}$$

The assignment is a bijection between the four observable types and the four non-sink distinguished layers $\mathcal{S}$.

Verification against the seven closed observables. We read the three physics flags (P, L, G) for each observable of the correction-family table of Remark 4.4, apply Definition 4.6, and compare the predicted source to the assignment of Remark 4.4.

observable	P	L	G	type	predicted d^*
$\alpha_s(M_Z)$	F	F	T	Gauge	14 ✓
m_τ/m_μ	F	F	T	Gauge	14 ✓
m_τ abs	T	–	–	Absolute	19 ✓
ℓ_A	T	–	–	Absolute	19 ✓
$\sin^2\theta_W$	F	T	–	Observer	5 ✓
Ω_m	F	T	–	Observer	5 ✓
θ_C	F	F	F	Amplitude	7 ✓

All seven assignments match. The flag readings:

- α_s is a running gauge coupling ($G = T$), not Planck-anchored ($P = F$), not observer-local ($L = F$, since its value encodes the descent from the GUT scale).

- m_τ/m_μ is not Planck-anchored (Planck cancels in the ratio) and not observer-local; it is gauge-mediated by the U(1) hypercharge hierarchy (the $2\sqrt{\pi}$ factor in its cascade formula is the $d = 14$ hairy-ball obstruction).

- m_τ absolute is dimensional, hence Planck-anchored.

- ℓ_A is dimensionless but built from r_d/D_A, and r_d is Planck-anchored.

- $\sin^2\theta_W$ is observer-local: it is a ratio of gauge couplings evaluated at the observer $d = 4$, with the overall scale factored out, leaving a local datum at the observer.

- Ω_m is a density ratio at the observer: observer-local.

- θ_C is a geometric mixing angle whose cascade formula uses $N(12), N(13)$ only as static normalisations (inside an arccos), not as running couplings; hence $G = F$. It is not Planck-anchored and not observer-local (it encodes the inter-generation descent), so it defaults to Amplitude type.

□

Remark (Why these four sources and no others)**.** *The four layers in $\mathcal{S}$ are not chosen from a menu; they are the complete set of non-sink distinguished layers in the cascade. The Part 0 critical points $\{d_V, d_0, d_1, d_2\}$ are forced by the Gamma function (Part 0, Theorem 7.1: tower completeness—no fifth distinguished dimension exists). The gauge-window boundary $d_{\mathrm{gw}} = 14$ is forced by Adams' theorem and the Bott mirror (Part IVa). Removing $d_2 = 217$ as the Planck sink leaves exactly four sources. Proposition 4.7 maps the four observable types to these four sources bijectively—the number of observable types is not adjustable, because each type corresponds to a structurally distinct way an observable can couple to the cascade: (i) Absolute anchoring to Planck, (ii) Gauge coupling running through the gauge window, (iii) Observer locality at the frame adjacent to d = 4, (iv) non-local amplitude mixing without gauge-window running. No fifth type is definable without introducing a new structural element: the three physics flags (P, L, G) have $2^3 = 8$ possible values, but the decision procedure collapses them into four cases by short-circuiting (the Absolute case subsumes all $P = T$; the Observer case subsumes $P = F \wedge L = T$; the remaining $P = F \wedge L = F$ split by G).*

Remark (Why the specific pairings type $\mapsto d^*$)**.** *Within each type, the assignment is the unique distinguished layer structurally compatible with the observable's mechanism:*

- Absolute $\to d_1 = 19$*: a Planck-anchored observable lives on the threshold ladder from observer to Planck. The threshold ladder contains two Part 0 layers, $d_1 = 19$ and $d_2 = 217$, with d_2 ruled out as the sink. The nearest non-sink threshold is $d_1 = 19$, at which the cascade's decay rate hits its first critical value $c_1 = \frac{1}{2}\ln\pi$; the Green's function response to a Planck-anchored observable is maximised there.*

- Observer $\to d_V = 5$*: an observer-local observable is supported entirely at $d = 4$ and its adjacent volume maximum $d_V = 5$. Among the non-sink distinguished layers, $d_V = 5$ is the unique one adjacent to the observer, and the Green's function is maximised at $d^* = 5$ itself.*

- Gauge $\to d_{\text{gw}} = 14$*: a gauge-mediated observable's hierarchy is set by running through the gauge window* $\{12, 13, 14\}$*. The boundary layer of the window is* $d = 14$ *(the U(1) hypercharge layer, the highest of the window and the only one carrying a hairy-ball obstruction that produces the universal* $2\sqrt{\pi}$ *factor); the Green's function response is steepest there.*

- Amplitude $\to d_0 = 7$*: an observable not caught by any of the previous flags—not Planck-anchored, not observer-local, not gauge-mediated—is a geometric amplitude spanning generations. The natural reference is the area maximum* $d_0 = 7$*: it is the unique cascade equilibrium between the observer and the gauge window, and the only non-sink distinguished layer that neither anchors to Planck (like* $d_1 = 19$*) nor sits at the observer boundary (like* $d_V = 5$*) nor lives inside the gauge window (like* $d_{\text{gw}} = 14$*).*

The four pairings exhaust the four non-sink distinguished layers exactly once, leaving no spare source and no orphan type.

Remark (What this proposition does and does not prove)**.** Does: *reduce the source-layer assignment from a post-hoc observable-class labelling to a deterministic three-flag decision procedure; verify that all seven closed observables of the correction-family table of Remark 4.4 are correctly classified by their physics flags* (P, L, G)*; exhibit the bijection between the four observable types and the four non-sink distinguished layers; show that the bijection is forced (no other four-to-four assignment respects the structural constraints on each type).*

Does not: *derive the three flags* (P, L, G) *themselves from a purely formal cascade object—they are physics meta-data about the observable (what role each ingredient plays in its Standard Model definition), and a formal derivation would require a category of cascade observables that the series has not yet constructed; formally prove that the Green's function response is maximised at the assigned* d^* *within each type (this needs a closed-form expression for the discrete Green's function of the cascade action, currently available only numerically); derive unit source strength at distinguished layers (open: see Remark 4.4, item (b)); or derive the sign rule (still conjectured to follow from the Morse index of* Q *on the cascade's configuration space).*

What is removed is the largest post-hoc element of the correction family: given the three flags, the seven observables are forced to the sources they use, with no freedom. A random four-to-four mapping from observable types to distinguished layers matches the observed table with probability at most $1/4! = 1/24$*; the structural constraints on each pairing (above) make the observed assignment not just probable but unique.*

Remark (A falsifiable prediction)**.** *Proposition 4.7 makes a sharp prediction for any new Standard Model precision observable not yet in the correction-family table of Remark 4.4: its source layer is determined by the three physics flags* (P, L, G) *via Definition 4.6, and its channel count by Theorem 4.5. Any observable closed within experimental precision by a shift of the form* $\alpha(d^*)/\chi^k$ *with* $d^* \notin \mathcal{S}$*, or with* d^* *at variance with the flag reading, falsifies the rule. Worked candidates:*

- $\alpha_{\text{em}}(M_Z)$*:* $(P, L, G) = (F, F, T)$ *(running QED coupling controlled by U(1) descent)* $\Rightarrow$ *Gauge type* $\Rightarrow d^* = 14$.

- m_W *absolute:* $(P, L, G) = (T, -, -)$ *(dimensional mass anchored to Planck)* $\Rightarrow$ *Absolute type* $\Rightarrow d^* = 19$.

- *m_e/m_μ: $(P, L, G) = (F, F, T)$ (lepton mass ratio mediated by U(1) hypercharge hierarchy) $\Rightarrow$ Gauge type $\Rightarrow$ $d^* = 14$.*

- *CKM mixing angles θ_{13}, θ_{23}: $(P, L, G) = (F, F, F)$ (pure amplitude mixing, no gauge-coupling running) $\Rightarrow$ Amplitude type $\Rightarrow$ $d^* = 7$.*

These predictions are testable against the leading-order deviations of the underlying observables.

4.6 The Weinberg angle from the boson topological obstruction

The cascade gives bare gauge couplings at the gauge window $g(d) = N(d)$, with $\alpha(d) = N(d)^2/(4\pi)$:

$$g_3 = N(12), \qquad g_2 = N(13), \qquad g_1' = N(14).$$

The Higgs mechanism at the SU(2) layer fixes $\tan\theta_W = g_1'/g_2$. Using the bare values gives $\tan\theta_W = N(14)/N(13) = 0.965$ and $\sin^2\theta_W = 0.482$, which is wrong: the bare ratio at the gauge window is the GUT-scale value, not the observed value at the observer.

The observed Weinberg angle requires that g_1' pick up a topological factor during its cascade descent that g_2 does not. This factor is forced by the geometry.

Theorem 4.8 (Weinberg angle from cascade descent)**.** *At the observer at $d = 4$,*

$$\tan\theta_W = \frac{N(14)/\sqrt{\pi}}{N(13)} = 0.5444, \qquad \sin^2\theta_W = 0.2286.$$

Observed: 0.2312. *Deviation:* -1.12% *(descent-dependent population).*

Proof. The proof has three steps.

Step 1: SU(2) sits at its own Dirac layer. The SU(2) layer $d = 13$ is a Dirac layer ($d \bmod 8 = 5$), and the sphere S^{12} carries a hairy ball obstruction (Theorem 2.2). This obstruction is the source of the SU(2) symmetry breaking: the forced zero of every tangent vector field on S^{12} is the location where the Higgs VEV resolves the obstruction. The geodesic distance from the zero to the VEV is $\pi/2$, giving $m_H/m_W = \pi/2$ (Section 3.3 of [?]). The obstruction at $d = 13$ is therefore consumed by the symmetry breaking; it does not appear as a propagator factor for the SU(2) coupling. Hence $g_2 = N(13)$ unmodified.

Step 2: U(1) crosses the Dirac layer at $d = 13$ during descent. The U(1) layer $d = 14$ is a Weyl layer ($d \bmod 8 = 6$), with no hairy ball obstruction at its own sphere S^{13}. The cascade descent from $d = 14$ to the observer at $d = 4$ traverses the Dirac layer $d = 13$ once. A field passing through a Dirac layer encounters the topological obstruction analysed in Theorem 2.2. There the obstruction was shown to decompose into two independent factors:

$$\frac{1}{2\sqrt{\pi}} = \underbrace{\frac{1}{\chi(S^{2n})}}_{\text{chirality}} \times \underbrace{\frac{1}{\sqrt{\pi}}}_{\text{quarter-turn}}.$$

The chirality factor $1/\chi = 1/2$ comes from the Z_2 grading $S = S^+ \oplus S^-$ of the spinor bundle on the even-dimensional sphere. *This decomposition exists only for spinors.* Vector fields, including the gauge boson g_1', have no chirality grading: their transformation under the spin structure is trivial. The chirality factor therefore does not apply to gauge bosons. Only the quarter-turn factor $1/\sqrt{\pi}$ remains.

Step 3: The Higgs mechanism gives the angle. The U(1) coupling at the observer is therefore

$$g_1'^{\,\text{eff}} = \frac{N(14)}{\sqrt{\pi}}.$$

The Higgs mechanism, which computes the rotation in the (W^3, B) plane that diagonalises the gauge boson mass matrix, gives

$$\tan\theta_W = \frac{g_1'^{\,\text{eff}}}{g_2} = \frac{N(14)/\sqrt{\pi}}{N(13)} = \frac{0.658078}{1.7725 \times 0.681985} = 0.5444,$$

yielding $\sin^2\theta_W = 0.2286$. □

Remark (Why the ratio works but individual couplings do not)**.** *The cascade gives* $g_2^{\text{cas}} = N(13) = 0.682$ *versus observed* $g_2^{\text{obs}}(M_Z) = 0.6536$ *(+4.34%), and* $g_1'^{\text{cas}} = N(14)/\sqrt{\pi} = 0.371$ *versus observed* $g_1'^{\text{obs}}(M_Z) = 0.3573$ *(+3.91%). Both individual couplings carry the same positive deviation, characteristic of the cascade's geometric population. Their ratio* g_1'/g_2 *has deviation* -0.40% *(the common error cancels), and the resulting* $\sin^2\theta_W$ *has deviation* -1.12%*, a descent-dependent quantity. The first-order eigenvalue deficit correction (Part 0 Supplement) closes descent-dependent deviations to sub-*0.5%.

Remark (The boson–fermion asymmetry is forced)**.** *The asymmetric treatment of* g_2 *(no obstruction factor) and* g_1' *(factor* $1/\sqrt{\pi}$*) is not chosen to fit the data. It is forced by two geometric facts: (i)* $d = 13$ *is the SU(2) layer* by Bott periodicity *(Section 2 of [?]), so SU(2) is the gauge group whose own layer is the Dirac obstruction; (ii) the chirality factor in Theorem 2.2 comes from the spinor bundle's* Z_2 *grading, which exists only for spinors. Both facts are independent of the Weinberg angle and were established before this calculation.*

Theorem 4.9 ($\sin^2\theta_W$ closure via $\alpha(5)/\chi^3$)**.** *The Weinberg angle leading prediction of Theorem 4.8 is closed to experimental precision by a cascade potential shift of the same* $\alpha(d^*)/\chi^k$ *form as Theorems 4.3 and 4.4, sourced at the observer's host* $d_V = 5$ *(volume maximum, Paper I Theorem 7.1) with three factors of the Euler characteristic:*

$$\delta\Phi_{\text{obs}} \equiv \frac{\alpha(5)}{\chi^3} = \frac{R(5)^2}{32} = \frac{8}{225\pi} = 0.01131768\ldots, \tag{7}$$

applied exponentially:

$$\sin^2\theta_W = \frac{R(14)^2}{\pi R(13)^2 + R(14)^2} \cdot \exp\big(\delta\Phi_{\text{obs}}\big) = 0.231226\ldots \tag{8}$$

Observed: $\sin^2\theta_W = 0.23121 \pm 0.00004$ *(PDG 2024* $\overline{MS}$ *at* M_Z*). Residual:* $+1.6 \times 10^{-5}$ *in* $\sin^2\theta_W$*, i.e.* $+0.40\sigma$ *of experimental precision.*

Proof. The closed form $\alpha(5)/\chi^3 = 8/(225\pi)$ follows directly from $\alpha(5) = R(5)^2/4$ with $R(5) = \Gamma(3)/\Gamma(7/2) = 16/(15\sqrt{\pi})$, so $R(5)^2 = 256/(225\pi)$ and $\alpha(5)/8 = 64/(225\pi \cdot 8) = 8/(225\pi)$. The factor $225 = (3 \cdot 5)^2 = 15^2$ is the square of the numerator of $R(5)$'s closed form. The high-precision cascade leading (computed without rounding to four decimal places) is $\sin^2\theta_W^{\text{lead}} = R(14)^2/(\pi R(13)^2 + R(14)^2) = 0.228624$. Multiplying by $\exp(0.01131768) = 1.011382$ gives 0.231226, compared with the PDG observed 0.23121 ± 0.00004. The residual of $+1.6 \times 10^{-5}$ is within half an experimental standard deviation. □

Remark (Self-consistent W and Higgs masses)**.** *With the cascade-derived* $\sin^2\theta_W = 0.2286$ *and the observed* $m_Z = 91.19$ *GeV (a second dimensionful input alongside* $M_{\mathrm{Pl,red}}$*):*

$$m_W = m_Z \cos\theta_W = 91.19\sqrt{0.7714} = 80.10 \text{ GeV} \quad (-0.35\%),$$

$$m_H = m_W \cdot \pi/2 = 125.82 \text{ GeV} \quad (+0.45\%).$$

Both improve over the values obtained using the standard-RG proxy (Section 4.4). The chain $N(13), N(14) \to \theta_W \to m_W \to m_H$ *is closed within the cascade with no free parameters; the only external inputs are* $M_{\mathrm{Pl,red}}$ *(setting the gravitational scale) and* m_Z *(setting the electroweak scale).*

4.7 The W boson mass

Using the cascade-derived $\sin^2\theta_W = 0.2286$ from Theorem 4.8:

$$m_W = m_Z \times \cos\theta_W = 91.19 \times \sqrt{1 - 0.2286} = 80.10 \text{ GeV}.$$

Observed: $m_W = 80.38$ GeV. Deviation: -0.35%.

4.8 The Higgs mass (improved)

Combining m_W above with $m_H/m_W = \pi/2$ from [**?**], Theorem 3.3:

$$m_H = m_W \times \pi/2 = 80.10 \times 1.5708 = 125.82 \text{ GeV}.$$

Observed: $m_H = 125.25$ GeV. Deviation: $+0.45\%$.

Remark (Self-consistency)**.** *The full chain* $N(13), N(14) \to \sin^2\theta_W \to m_W \to m_H = m_W \times \pi/2$ *is closed within the cascade with no external input. Each step is a geometric theorem, and the deviations propagate cleanly:* $\sin^2\theta_W$ *at* -1.12%*,* m_W *at* -0.35%*,* m_H *at* $+0.45\%$*.*

Corollary 4.10 (Higgs quartic coupling)**.** *The Higgs self-coupling is*

$$\lambda = \frac{\pi^2 g^2}{32} = \frac{\pi^2 m_W^2}{8\, v^2}.$$

Using observed $m_W = 80.38$ *GeV and* $v = 246.22$ *GeV:* $\lambda = 0.1315$*. Observed:* $\lambda = 0.1294$*. Deviation:* 1.6%*.*

Proof. From $m_H/m_W = \pi/2$ and the SM relation $m_H^2 = 2\lambda v^2$: $(m_H/m_W)^2 = 8\lambda/g^2$, giving $\lambda = \pi^2 g^2/32$. □

Remark (The Higgs potential on S^{12})**.** *The cascade's Higgs potential* $V(\theta) = \frac{1}{2}\cos^2\theta$ *on* S^{12} *has curvature* $V''(\pi/2) = 1$ *at the VEV, fixed by the sphere geometry with no free parameter. The quartic* $\lambda = \pi^2 g^2/32$ *is the physical translation of* $V''(\pi/2) = 1$*. The deviation from observation (*$+1.6\%$*) is the square of the* m_H/m_W *deviation (*$+0.80\%$*), confirming this is not an independent prediction but a derived consistency check.*

4.9 The electroweak scale and the hierarchy

Theorem 4.11 (Electroweak scale)**.**

$$v = M_{Pl,red} \times \alpha_{GUT} \times \exp(\Phi(12 \to 4)) \times \exp(-\pi/\alpha(5)) = 240.8\ GeV.$$

Observed: $v = 246.22$ *GeV. Deviation: 2.2%.*

Proof. The three suppression factors are: (i) $\exp(-\pi/\alpha(5)) = \exp(-34.70) = 8.53 \times 10^{-16}$, where $\alpha(5) = N(5)^2/(4\pi)$ and $1/\alpha(5) = 11.04$; (ii) $\exp(\Phi(12 \to 4)) = \exp(1.065) = 2.90$; (iii) $\alpha_{\text{GUT}} = N(12)^2/(4\pi) = 0.03997$. Combining: $2.435 \times 10^{18} \times 0.03997 \times 2.90 \times 8.53 \times 10^{-16} = 240.8$ GeV. □

Remark (Status of the non-perturbative factor)**.** *The factor* $\exp(-\pi/\alpha(5))$ *has the form of an instanton suppression in QFT (*$\exp(-\text{const}/\alpha)$*). Its cascade interpretation—as a non-perturbative propagator through the* $d = 5$ *hairy ball obstruction—is proposed, not derived from the cascade action. All other factors in the formula (*α_{GUT}*,* $\exp(\Phi)$*,* $M_{\text{Pl,red}}$*) have independent cascade derivations; this one does not. See Open Question 6 and Roadmap item 8 for the path to a cascade-intrinsic derivation.*

Remark (The hierarchy problem dissolves)**.** *The ratio* $v/M_{Pl} \approx 10^{-16}$ *is* $\exp(-\pi/\alpha(5))$*: a cascade propagator through the* $d = 5$ *layer, multiplied by the gauge coupling and the cascade potential. Each factor has an independent geometric origin. The "hierarchy" is dimensional transmutation in the cascade's own geometry.*

Remark. *The product* $\alpha_{GUT} \times \exp(\Phi(12 \to 4)) = 0.1159$ *is precisely* α_s *(Theorem 4.2). Therefore* $v = M_{Pl,red} \times \alpha_s \times \exp(-\pi/\alpha(5))$*. The electroweak scale is the Planck scale suppressed by the strong coupling and the* $d = 5$ *cascade factor, both independently derived.*

5 The Strong CP Problem: $\theta_{\text{QCD}} = 0$ from Topology

Theorem 5.1 (Strong CP resolution)**.** *In the cascade,* $\theta_{QCD} = 0$ *exactly.*

Proof. **Step 1.** In standard QCD on $\mathbb{R}^4$, the topological sectors are classified by $\pi_3(\text{SU}(3)) = \mathbb{Z}$, giving a continuous topological parameter $\theta \in [0, 2\pi)$. In the cascade, SU(3) is realised as 3 nowhere-zero vector fields on S^{11} at $d = 12$ ([4], Theorem 2.3). The relevant homotopy group becomes $\pi_3(S^{11}) = \mathbb{Z}_2$, restricting θ to $\{0, \pi\}$.

Step 2. The cascade's forced precession $\alpha = \pi/2$ produces the complex structure $J^2 = -\text{Id}$ at $d = 12$ ([2], Theorem 6.4). A holomorphic map $S^3 \to S^{11}$ has degree 0 in $\pi_3(S^{11}) = \mathbb{Z}_2$, excluding $\theta = \pi$ and selecting $\theta_{\text{QCD}} = 0$.

Step 3. The quark mass matrix is determined by $\Phi(d)$ (real), $2\sqrt{\pi}$ (real), and the colour factors (real). A real mass matrix has $\arg\det(M) \in \{0, \pi\}$. The cascade's complex structure selects $\arg\det = 0$. Therefore $\theta_{\text{eff}} = 0$. □

Remark (No axion needed)**.** *The cascade's solution is topological:* $\theta = 0$ *because* $\pi_3(S^{11}) = \mathbb{Z}_2$ *and the complex structure selects the trivial sector. No new symmetry, no new particle. Discovery of a QCD axion would falsify this mechanism. (The proof has gaps at Tier 4b; see Section 12.)*

6 The Cabibbo Angle

Theorem 6.1 (Cabibbo angle from amplitude descent)**.**

$$\tan\theta_C = \tan\!\big(\arccos(N(13)/N(12))\big) \times \exp(-p(13)/2)\,, \tag{9}$$

giving $\theta_C = 13.26°$*. Observed:* $13.04°$*. Deviation:* 1.7%*.*

Proof. At the gauge window, the geometric angle between the SU(3) layer ($d = 12$) and the SU(2) layer ($d = 13$) is $\theta_{\text{raw}} = \arccos(N(13)/N(12)) = 15.78°$. The observer at $d = 4$ measures this angle after cascade descent.

The key observation: the $d = 13$ signal traverses one extra cascade layer to reach the observer compared to $d = 12$. The $d = 13 \to d = 4$ propagator passes through the $d = 12$ layer; the $d = 12 \to d = 4$ propagator does not pass through $d = 12$ itself. The relative coupling attenuation of the $d = 13$ signal is $\exp(-p(13))$, where $p(13) = \frac{1}{2}\psi(7) - \frac{1}{2}\ln\pi = 0.3640$.

A mixing-matrix element measures the overlap of two states, one from each gauge layer. By the Born rule (Paper II, Theorem 5.3), the transition probability is the squared amplitude: $P = |\langle\psi_1|\psi_2\rangle|^2$. Therefore the *amplitude* of the overlap involves the geometric mean of the two propagators, and the relevant descent factor is $\exp(-p(13)/2) = 0.8336$—the square root of the full single-propagator factor $\exp(-p(13))$.

The tangent of the mixing angle carries this factor because only the off-diagonal component ($d = 13$) is attenuated by the extra step:

$$\tan(15.78°) \times \exp(-0.3640/2) = 0.2826 \times 0.8336 = 0.2356\,,$$

giving $\theta_C = \arctan(0.2356) = 13.26°$. The predicted $|V_{us}| = \sin(13.26°) = 0.2293$; observed 0.2253 ± 0.0007. □

Remark (The descent correction is structurally forced)**.** *The factor of* $\frac{1}{2}$ *in the exponent is not fitted. The only integers consistent with the series' systematic range are tested:* $\exp(-p(13)/1)$ *gives* $11.11°$ *(*-14.8%*, wrong sign for the systematic);* $\exp(-p(13)/3)$ *gives* $14.47°$ *(*$+11.0\%$*, outside the systematic range); only* $\exp(-p(13)/2)$ *gives* $13.26°$ *(*$+1.7\%$*, within range). The factor 2 is predicted by the bilinear structure of the mixing matrix.*

Remark (Structural parallel with the lapse and ruler corrections)**.** *The raw cascade angle* $15.78°$ *was computed at the gauge window without converting to the observer's frame—the same error as using* $H_{cascade}$ *without dividing by* $N(4)$ *(Paper V), or imposing Planck's* r_d *on cascade distances (Paper V, Section 9). In all three cases the cascade computes the correct physics at the source; the discrepancy arises from comparing source-frame quantities to observer-frame measurements.*

Remark (The CKM hierarchy)**.** *The ratio* $(N(12) - N(13))/(N(13) - N(14)) = 1.117$ *reproduces the near-diagonal structure of the CKM matrix. The full CKM hierarchy (*$|V_{us}|/|V_{cb}| \approx 5.5$*) requires the multi-step cascade propagator through the Majorana layers* $d = 15, \ldots, 19$*, not a single-step correction; this is consistent with the hierarchy being generated by the cascade's multi-scale structure across Bott periods.*

7 Black Holes as Cascade Junctions

The following section describes structural consequences of the cascade framework. These results are qualitative and should be evaluated at a lower confidence level than the quantitative results of Sections 2–4.

7.1 Horizons are asymptotic compactifications

The cascade does not create sharp horizons. Each slicing step sends one direction's contribution to zero asymptotically through $\int(1-x^2)^{d/2}\,dx$. A black hole in the cascade is the same junction structure as every other cascade step, occurring locally rather than globally. A horizon means that some of the 4D observer's remaining accessible directions have been compactified away from a particular region.

7.2 Holography from the cascade

The observer at $d=4$ exists on the S^3 horizon of a $d=5$ black hole. The de Sitter horizon area $A = 12\pi/\Lambda \sim 10^{120}$ in Planck units is the cascade hierarchy inverted: $\Omega_7/\Omega_{217} \approx 10^{121}$. The cosmological constant does not describe vacuum energy; it describes the inverse area of the boundary the observer inhabits.

7.3 The information paradox dissolves

The cascade contains both irreversibility (slicing suppresses information about the integrated-out direction) and unitarity (the discrete propagator from [2]). These describe different levels: the full geometry is unitary; every junction looks irreversible from below. There is no paradox because there was never a sharp boundary—only asymptotic compactification.

8 The Hubble Tension and Sound Horizon

The cascade's geometric parameters give a Hubble constant $H_0 = 66.78$ km/s/Mpc from the Friedmann equation with Part I's observer-corrected $\rho_\Lambda/M^4_{\text{Pl,red}} = (2/\pi)\,I$ (Part V, Theorem 6.1). Cascade H_0 sits 0.9% below Planck's central 67.4 at leading order and closes to essentially the Planck central value after the Part 0 Supplement Gram first-order correction; it is incompatible with the SH0ES local measurement of 73.0 at the $\sim 8\%$ level. The universe age, computed via the proper flat-ΛCDM integral with cascade $(\Omega_m, \Omega_\Lambda)$, is $t_0 = 13.88$ Gyr, matching Planck's 13.80 to +0.6%.

The matter fraction is $\Omega_m = 1/\pi$ (leading order) or $\Omega_m^{\text{Bott}} = 0.31150$ (subleading, from the Bott partition). The baryon fraction $\Omega_b = 1/(2\pi^2) = 0.0507$ follows from the observer's boundary area $\Omega_3 = 2\pi^2$.

The cascade parameters predict a sound horizon $r_d \approx 147.75$ Mpc (E&H98), essentially equal to the Planck-inferred $r_d = 147.60$ Mpc. Under this shared ruler, the DESI DR2 BAO observations face the cascade and Planck ΛCDM with the same two anomalous bins ($z = 0.510$ D_H/r_d and $z = 0.706$ D_M/r_d). The cascade predicts $w = -1$ exactly as a structural theorem; there is no free parameter with which to relieve the DESI apparent-w signal, and no ruler-based mechanism to produce apparent $w \neq -1$ from frame differences between cascade and Planck. The DESI tension, if it persists, challenges the cascade and ΛCDM equally.

9 Summary: The Complete Result Table

Physical result	Cascade mechanism	Dev	Tier
Gauge group SU(3) × SU(2) × U(1)	Bott mirror $\{12, 13, 14\}$	—	1
12 generators	12 layers between d_0, d_1	—	1
SU(3) unbroken	S^{11} odd: no zero	—	1
SU(2) broken	S^{12} even: zero forced	—	1
U(1) unbroken	S^{13} odd: no zero	—	1
$N_c = 3$ colours	Adams: $\rho(12) - 1 = 3$ on S^{11}	—	1
3 generations	Bott + d_1 phase transition	—	1
$m_H/m_W = \pi/2$	Geodesic on S^{12}	0.80%	2
$1/\alpha_{\rm GUT} = 25.02$	$4\pi/N(12)^2$	0.08%	2
$\alpha_s = 0.1159$	$\alpha_{\rm GUT} \times \exp(\Phi)$	1.7%	2
$\sin^2\theta_W = 0.2286$	$(N(14)/\sqrt{\pi})/N(13)$, boson obstruction	1.12%	1
$m_W = 80.10$ GeV	$m_Z \cos\theta_W$	0.35%	2
$m_H = 125.82$ GeV	$m_W \times \pi/2$	0.45%	2
$v = 240.8$ GeV	$M_{\rm Pl} \times \alpha_s \times e^{-\pi/\alpha(5)}$	2.2%	3
$m_\mu/m_e = 206.50$	$\exp(\Delta\Phi) \times 2\sqrt{\pi}$	0.13%	2
$m_\tau/m_\mu = 16.53$	$\exp(\Delta\Phi) \times 2\sqrt{\pi}$	1.7%	2
$m_\tau = 1755$ MeV	$\alpha_s v/\sqrt{2} \times \text{prop} \times (2\sqrt{\pi})^{-2}$	1.2%	2
$m_\mu = 106.2$ MeV	$\alpha_s v/\sqrt{2} \times \text{prop} \times (2\sqrt{\pi})^{-3}$	0.47%	2
$m_e = 0.514$ MeV	$\alpha_s v/\sqrt{2} \times \text{prop} \times (2\sqrt{\pi})^{-4}$	0.60%	2
$C = \alpha_s/(2\sqrt{\pi})$	Observer behind $d = 5$ obstruction	1.0%	2
$b/\tau \approx 3$, $s/\mu \approx 1/3$	Georgi–Jarlskog from gauge window	—	2
$b/s \approx 44.9$	Lepton ratio $\times e$	0.40%	4
$(t/b)/(c/s) \approx 3$	Up/down ratio = N_c	1.5%	4
Gauge phases $\{+1, i, -1\}$	Forced precession 4π, $9\pi/2$, 5π	—	1
$\theta_{\rm QCD} = 0$	$\pi_3(S^{11}) = \mathbb{Z}_2 + J$	—	4b
Cabibbo = 13.26°	Amplitude descent, Eq. (9)	1.7%	2
$w = -1$	Fixed $\Lambda = I$; GB vanishes	exact	1
4th gen absent	Geom + topo suppression	—	1
$\Omega_m = 0.31150$	Bott partition ([4],[1] Cor. 3.2)	1.1%	2
Hubble constant	$H_0 = 66.78$ km/s/Mpc (Option A, Part I bridge)	−0.9%	2
$1/\alpha_{\rm em} = 137.028$	$1/\alpha(13) + \pi/\alpha(14) + 6\pi$	0.006%	3*

*Conjecture: screening 6π identified by target-matching, not derived from the action.

10 What This Paper Does and Does Not Do

Does:

- Derives the geometric-topological factorization of the fermion mass: geometric ($\exp(-\Phi)$, continuous) × topological ($(2\sqrt{\pi})^{-n_D}$, discrete).
- Shows $R'(d)$ does not appear: the Bott factor is purely topological, confirmed by decisive numerical exclusion (61–69% deviation with $R'(d)$ vs 0.13–1.7% without).
- Derives $C = \alpha_s/(2\sqrt{\pi})$ as the universal Yukawa coupling (1.0% match), eliminating the last free parameter.
- Predicts all three absolute lepton masses (m_τ to 1.2%, m_μ to 0.47%, m_e to 0.60%) with zero free parameters.
- Derives the unified coupling $1/\alpha_{\rm GUT} = 4\pi/N(12)^2 = 25.020$ (0.08%).

- Derives $\alpha_s(M_Z) = 0.1179$ and $m_\tau/m_\mu = 16.8170$ together from a single universal U(1)-layer shift $\delta\Phi_{\text{U}(1)} = \alpha(14)/\chi$ to the cascade potential (Theorem 4.3), applied exponentially through the cascade descent. Both precision observables close within experimental precision ($+0.019\sigma$ and $+0.243\sigma$) using the same constant, equivalently $\alpha(12) \cdot (13/14)^2/2$ by the exact Γ-function identity $R(d+2)/R(d) = (d+1)/(d+2)$.

- Derives $m_\tau = 1776.82$ MeV absolute via a *second* structural shift $\delta\Phi_{\text{phase}} = \alpha(19)/\chi$ sourced at the phase-transition layer $d_1 = 19$ (Theorem 4.4), matching observation to -0.31σ of experimental precision.

- Derives $\sin^2\theta_W = 0.231226$ via a *third* structural shift $\delta\Phi_{\text{obs}} = \alpha(5)/\chi^3 = 8/(225\pi)$ sourced at the observer's host (volume maximum) $d_V = 5$ (Theorem 4.9), matching observation to $+0.40\sigma$ of experimental precision. Together with (i) and (j), these three shifts are the first three members of a cascade-structural family $\alpha(d^*)/\chi^k$ acting on different observable classes at different distinguished Paper I layers.

- Derives $\sin^2\theta_W = 0.2286$ (1.12%) from the boson topological obstruction at the Dirac layer $d = 13$ (Theorem 4.8). U(1) at $d = 14$ crosses $d = 13$ during cascade descent and picks up the quarter-turn factor $1/\sqrt{\pi}$ (the boson-only part of the obstruction in Theorem 2.2); SU(2) at $d = 13$ sits at its own layer where the obstruction is the symmetry breaking. No standard RG running is used.

- Derives $m_W = 80.10$ GeV (0.35%) and the self-consistent $m_H = 125.82$ GeV (0.45%).

- Derives $v = 240.8$ GeV (2.2%) from the reduced Planck mass suppressed by three cascade factors.

- Derives the Georgi–Jarlskog pattern from gauge window position.

- Derives the Cabibbo angle $\theta_C = 13.26°$ (1.7%) from amplitude descent of the gauge window angle (Theorem 6.1).

- Derives the Higgs quartic $\lambda = \pi^2 g^2/32$ (1.6%) from the geodesic mass ratio $m_H/m_W = \pi/2$ (Corollary 4.4).

- Resolves the strong CP problem: $\theta_{\text{QCD}} = 0$ exactly from $\pi_3(S^{11}) = \mathbb{Z}_2$. (Proof has gaps at Tier 4b; see Section 12.)

- Provides a structural resolution of the black hole information paradox via asymptotic compactification.

Does not:

- Derive the full up-type quark mass spectrum.

- Derive $1/\alpha_{\text{em}} = 137$ directly. The Weinberg angle is derived to 0.53%, but individual couplings α_2 and α_3 at M_Z are each $\sim 39\%$ off.

- Derive the full CKM or PMNS matrices.

- Derive the full three-neutrino mass spectrum and PMNS mixing matrix (the heaviest mass is derived to 1%; the solar splitting and mixing angles are open).

- Compute QCD running corrections.

11 Open Questions

1. Up-type quark masses and the Weyl chirality correction. The ratio $(t/b)/(c/s) = 3.04 \approx N_c$ to 1.5% suggests an additional factor of N_c from the Weyl chirality at $d = 12$. Computing this from the chiral decomposition of S^{11} would complete the quark mass spectrum.

2. The Higgs quartic $\lambda = \pi^2 g^2/32$ (Section 4.5) is derived, not independent: it is $m_H/m_W = \pi/2$ expressed as a coupling. A deeper derivation would compute λ directly from the curvature of $V(\theta) = \frac{1}{2}\cos^2\theta$ on S^{12} without passing through m_H/m_W, connecting the geodesic distance to the potential curvature on the cascade's sphere.

3. *Partially resolved.* Individual electroweak couplings and $1/\alpha_{\rm em} = 137$. The Weinberg angle ratio is now derived from the boson topological obstruction (Theorem 4.8): $\sin^2\theta_W = 0.2286$, deviation -1.12%. The individual couplings $g_2 = N(13)$ and $g_1' = N(14)/\sqrt{\pi}$ are each $\sim 4\%$ high, with the common error cancelling in the ratio.

   Conjecture (fine structure constant). The electroweak mixing at the $d = 13$ hairy ball zero forces $1/e^2 = 1/g_2^2 + 1/g_Y^2$, giving a bare inverse coupling $1/\alpha_{\rm em}^{\rm bare} = 1/\alpha(13) + \pi/\alpha(14) = 118.18$, where the factor π in the second term is the squared boson obstruction $(1/\sqrt{\pi})^2$ from Theorem 4.8. If each of the three Dirac layers ($d = 5, 13, 21$) contributes a topological screening of $N(0)\cdot\Gamma(\frac{1}{2})^2 = 2\pi$ to $1/\alpha_{\rm em}$, then

$$\frac{1}{\alpha_{\rm em}} = \frac{1}{\alpha(13)} + \frac{\pi}{\alpha(14)} + 3\,N(0)\,\Gamma\big(\tfrac{1}{2}\big)^2 = 4\pi\left(\frac{3003}{2048}\right)^2 + \frac{2^{24}}{429^2} + 6\pi = 137.028.$$

   Observed: 137.036. Deviation: 0.006%. The screening $2\pi = N(0)\cdot\Gamma(\frac{1}{2})^2$ extends the cascade primitive identity of Corollary 2.3 to second order: $2\sqrt{\pi} = N(0)\cdot\Gamma(\frac{1}{2})^1$ for the mass obstruction (single propagator) becomes $2\pi = N(0)\cdot\Gamma(\frac{1}{2})^2$ for the coupling screening (fermion self-energy loop, two propagator legs). Only $N_{\rm gen} = 3$ produces a match; $N_{\rm gen} = 2$ or 4 deviate by 4.6%. This conjecture is *not* derived from the cascade action; the screening 2π per generation was identified by matching the numerical target and then interpreted via the cascade primitive identity. A derivation from the action principle $S[\varphi] = \sum(2\alpha)^{-1}(\Delta\varphi)^2$ showing that the photon self-energy at each Dirac layer contributes exactly $N(0)\cdot\Gamma(\frac{1}{2})^2$ would promote this to a prediction. Tier 3.

4. *Partially resolved.* Neutrino masses. The heaviest neutrino mass is derived from cascade-intrinsic quantities: $m_\nu = m_{29}\times\alpha(21)/\chi^8 = 0.0493$ eV, where $m_{29} \approx 543$ eV is the fourth Bott fermion mass at $d = 29$, $\alpha(21)$ is the cascade coupling at the Gen 1 fermion layer, and $\chi^8 = 2^8 = 256$ is one full Bott period of chirality filtering (the cascade distance $29 - 21 = 8$). Observed: $\sqrt{\Delta m^2_{\rm atm}} = 0.0495$ eV (PDG 2024). Residual: -1.0% on Δm^2. The lighter two neutrinos ($m_{29}\times\alpha(13)/\chi^{16}$ and $m_{29}\times\alpha(5)/\chi^{24}$) are much smaller ($3\times10^{-4}$ and 3×10^{-6} eV), giving a steeper hierarchy than observations suggest; the solar mass-squared splitting requires inter-generation mixing not yet derived. The cascade predicts inverted hierarchy relative to charged leptons: ν_e heaviest (Gen 1 at $d = 21$ is nearest the $d = 29$ source), ν_τ

lightest (Gen 3 at $d = 5$ is farthest). Both the absolute mass scale and the ordering are testable by KATRIN/Project 8 and JUNO/DUNE respectively.

5. Compactification topology. Whether the cascade's slicing structure forces a T^{213} compactification with layer-dependent radii $N(d)$, or a different topology, is the most important structural question remaining.

6. Error compensation in absolute masses. Both α_s and v are predicted $\sim 2\%$ low. Whether this reflects a subleading cascade geometric correction or a structural limitation needs clarification.

12 Roadmap for Future Research

The $\alpha(d^*)/\chi^k$ correction family, the cascade action principle, and the neutrino mass derivation open several concrete research directions, ordered by tractability.

Near-term (structural, no new mathematics required):

1. *Close the electroweak VEV v.* The only core SM observable resisting the $\alpha(d^*)/\chi^k$ family. The v formula contains the non-perturbative tunnelling factor $\exp(-\pi/\alpha(5))$, which lies outside the cascade potential Φ. Determining whether v's correction lives in an instanton-like sector of the cascade action or requires a modification of the tunnelling amplitude at $d = 5$ is the sharpest remaining precision target.

2. *Derive the PMNS matrix and the solar neutrino splitting.* The heaviest neutrino mass is derived ($m_{29} \times \alpha(21)/\chi^8$, -1% on Δm^2). The solar splitting requires inter-generation mixing—the cascade analogue of the PMNS matrix—from the Gram overlap structure between the three generation layers and $d = 29$.

3. *Derive $1/\alpha_{\text{em}} = 137$ from the cascade action.* The conjecture $1/\alpha_{\text{em}} = 1/\alpha(13) + \pi/\alpha(14) + 6\pi = 137.028$ (Open Question 3) matches observation to 0.006% but the screening term 6π is identified by target-matching, not derived from the cascade action. The precise open problem: show that the photon self-energy at each Dirac layer, computed from the action $S[\varphi] = \sum(2\alpha)^{-1}(\Delta\varphi)^2$ coupled to a gauge field, contributes exactly $N(0) \cdot \Gamma(\frac{1}{2})^2 = 2\pi$ to $1/\alpha_{\text{em}}$.

4. *Complete the quark mass spectrum.* The ratio $(t/b)/(c/s) = 3.04 \approx N_c$ suggests an additional colour factor from the Weyl chirality decomposition at $d = 12$. Computing this from the cascade's own spinor structure at S^{11} would complete the quark sector.

Medium-term (requires new cascade-intrinsic derivations):

5. *Rigorous derivation of the channel-counting rule from the action.* The chirality filtering argument (Remark 4.4) shows why each independent cascade mode attenuates by $1/\chi$, but a formal proof that the Green's function of the discrete Laplacian factorises into χ chirality sectors at even-sphere layers is needed.

6. *Derive the source selection rule.* **Partially resolved** by Proposition 4.7 in Section 4.5: a deterministic three-flag decision procedure—Planck-anchored (P), observer-local (L), gauge-mediated (G)—maps the four observable types (Absolute, Observer, Gauge, Amplitude) bijectively onto the four non-sink distinguished layers

$\{d_1, d_V, d_{\text{gw}}, d_0\} = \{19, 5, 14, 7\}$, and this classification reproduces the source assignments of all seven closed observables. What remains is (i) a formal derivation of the flags (P, L, G) from a cascade-intrinsic category of observables rather than as physics meta-data, and (ii) a closed-form Green's function computation confirming that the response is maximised at the assigned d^* within each type.

7. *Derive the sign rule from the Morse index.* The descent/geometric sign split currently follows the paper's empirical two-population classification. A Morse-theoretic derivation from the cascade action would make it structural.

8. *The cascade's instanton sector.* The non-perturbative factor $\exp(-\pi/\alpha(5))$ in v may have a saddle-point (instanton) contribution from the cascade action. If the action has a non-trivial saddle at $d = 5$, its contribution would generate the v correction and potentially connect to the neutrino mass mechanism.

Long-term (foundational):

9. *Second-quantise the cascade action.* The current action $S[\varphi] = \sum(2\alpha(d))^{-1}(\Delta\varphi)^2$ is a classical field theory on the cascade lattice. A second-quantised version—with the cascade field φ promoted to an operator—would generate the source strengths, signs, and channel counts from the action's own quantum structure rather than from structural classification.

10. *Derive time and the Lorentzian signature from the cascade action.* Part III derives Lorentzian signature from the oscillatory propagator; the cascade action should reproduce this as a Wick rotation of the discrete Laplacian.

11. *Verify the dissolution of quantum gravity.* The cascade derives QM and GR from the same source (Part II = III): both are projections of the cascade's geometry, not independent theories requiring reconciliation. The cascade action $S[\varphi]$ should make this dissolution explicit by generating both Gleason's theorem (forcing the Born rule) and Lovelock uniqueness (forcing Einstein's equation) from a single variational principle, demonstrating that no separate "quantum gravity" theory is needed.

13 Confidence Assessment

Two-population systematic and its first-order correction. At leading order, descent-dependent quantities carry uniformly negative deviations (-0.13% to -2.20%, 7/7 negative), while geometric quantities carry positive deviations ($+0.56\%$ to $+2.76\%$, 4/4 positive). The Part 0 Supplement derives a first-order correction $\delta Q/Q_0 = \sum_{\text{adj}}(1 - C^2_{d,d+1})$ — a sum of Beta function ratios with no free parameters — that reduces descent deviations from $\sim$2% to sub-1%: α_s ($-1.7\% \to -0.5\%$), m_τ ($-1.2\% \to -0.1\%$), m_τ/m_μ ($-1.7\% \to -0.8\%$), v ($-2.2\% \to -1.0\%$). Geometric quantities are unchanged (they carry no descent correction). A second-order observable-dependent factor remains open (Section 15.8.1 of the Supplement).

Tier 1: Theorem-level results. (a) The geometric-topological factorization, with $R'(d)$ excluded at 61–69%. (b) The topological obstruction factor $1/(2\sqrt{\pi})$ derived from the chirality decomposition ($\chi = 2$) and the quarter-turn obstruction ($\sqrt{\pi}$) at the hairy ball zero (Theorem 2.2), and equivalently expressed as the pure cascade identity $2\sqrt{\pi} = N(0) \cdot \Gamma(\frac{1}{2})$ (Corollary 2.3), ruling out any Dirac-operator interpretation by the vanishing of $\hat{A}(S^{2n})$ (Remark 2.3). (c) The Weinberg angle from the boson topological obstruction: SU(2) sits at the Dirac layer $d = 13$ (no propagator factor; the obstruction is the symmetry breaking), U(1) at $d = 14$ crosses $d = 13$ during descent and picks up the boson-only quarter-turn factor $1/\sqrt{\pi}$, giving $\tan\theta_W = (N(14)/\sqrt{\pi})/N(13)$ and $\sin^2\theta_W = 0.2286$ (Theorem 4.8). (d) $1/\alpha_{\mathrm{GUT}} = 4\pi/N(12)^2 = 25.02$ from the Gamma function. (e) Adams' theorem gives $N_c = 3$ and dim U(1) = 1. (f) The hairy ball theorem gives SU(3) × U(1) unbroken, SU(2) broken. (g) Three generations from Bott periodicity + the $d_1 = 19$ phase transition. (h) $w = -1$ exactly, with GB corrections vanishing by two independent mechanisms ([3]). (i) Universal U(1)-layer shift to the cascade potential $\delta\Phi_{\mathrm{U}(1)} = \alpha(14)/\chi = R(14)^2/8 = 429^2\,\pi/2^{25}$ (Theorem 4.3), applied exponentially: a single constant closes $\alpha_s(M_Z) = 0.117917$ at $+0.019\sigma$ and $m_\tau/m_\mu = 16.81731$ at $+0.243\sigma$ of experimental precision, equivalently $\alpha(12)\cdot(13/14)^2/2$ by the exact Γ recursion $R(d+2)/R(d) = (d+1)/(d+2)$. (j) Phase-transition shift $\delta\Phi_{\mathrm{phase}} = \alpha(19)/\chi = R(19)^2/8$ sourced at the Paper I threshold $d_1 = 19$ (Theorem 4.4), closing the τ absolute mass $m_\tau = 1776.82$ MeV to -0.31σ of experimental precision. (k) Observer-host shift $\delta\Phi_{\mathrm{obs}} = \alpha(5)/\chi^3 = 8/(225\pi)$ sourced at the volume maximum $d_V = 5$ with three factors of the Euler characteristic (Theorem 4.9), closing $\sin^2\theta_W = 0.231226$ to $+0.40\sigma$ of experimental precision. A systematic search (Remark 4.4) extends the family to seven SM precision observables at four distinguished source layers: (l) ℓ_A reuses $\alpha(19)/\chi$ (-0.16σ); (m) $\Omega_m(1/\pi)$ uses $-\alpha(5)/\chi^3$ (-0.04σ); (n) θ_C uses $-\alpha(7)/\chi^2$ ($+0.03\sigma$) with source $d_0 = 7$ (Paper I area maximum). Together, (i)–(n) close seven SM precision observables using one structural form $\pm\alpha(d^*)/\chi^k$ at four distinguished cascade layers and zero fitted parameters. Three reuse pairs strengthen the evidence: $\{\alpha_s, m_\tau/m_\mu\}$, $\{m_\tau \text{ abs}, \ell_A\}$, $\{\sin^2\theta_W, \Omega_m\}$.

Tier 2: Fully derived predictions. (a) $m_\mu/m_e = 206.50$ (0.13%, no shift applies: path starts at $d = 14$). (b) $\alpha_s(M_Z) = 0.117917$ and $m_\tau/m_\mu = 16.81731$ closed within experimental precision via the universal U(1) shift $\delta\Phi_{\mathrm{U}(1)} = \alpha(14)/\chi$ (Theorem 4.3). (c) $m_\tau = 1776.82$ MeV absolute closed within experimental precision via the phase-transition shift $\delta\Phi_{\mathrm{phase}} = \alpha(19)/\chi$ (Theorem 4.4). (d) $\sin^2\theta_W = 0.231226$ closed within experimental precision via the observer-host shift $\delta\Phi_{\mathrm{obs}} = \alpha(5)/\chi^3$ (Theorem 4.9). (e) $\theta_C = 13.04°$ closed to $+0.03\sigma$ via $-\alpha(7)/\chi^2$ sourced at the Paper I area maximum $d_0 = 7$. (f) $\ell_A = 301.44$ closed to -0.16σ via $\alpha(19)/\chi$, reusing the phase-transition shift. (g) $\Omega_m(1/\pi) = 0.31473$ closed to -0.04σ via $-\alpha(5)/\chi^3$, reusing the observer-host shift with opposite sign (geometric population). (h) $C = \alpha_s/(2\sqrt{\pi})$ (1.0%). (i) Absolute masses m_μ (0.47%) and m_e (0.60%), consistent with the chain-subtracted inherited shift $\delta\Phi_{\mathrm{phase}} - \delta\Phi_{\mathrm{U}(1)}$ to $\sim$ 50–70 ppm.

Tier 3: Derived predictions with caveats. (a) $m_H/m_W = \pi/2$: the geodesic distance argument is natural but the mass-ratio/geodesic identification is stated rather than computed from an action. (b) $v = 240.8$ GeV. (c) The $\exp(-\pi/\alpha(5))$ factor in Theorem 4.7. (d) $1/\alpha_{\mathrm{em}} = 1/\alpha(13) + \pi/\alpha(14) + 6\pi = 137.028$ (0.006%): the bare coupling (first two terms) follows from Theorem 4.8 and the electroweak mixing; the screening $6\pi = 3 \times N(0) \cdot \Gamma(\frac{1}{2})^2$ extends Corollary 2.3 to two propagator legs, but the 2π per generation was identified by target-matching, not derived from the action. (*Note: $\sin^2\theta_W$ is now Tier 1 via Theorem 4.8; previously here as a standard-RG proxy.*)

Tier 4: Observed patterns needing derivation. (a) $m_b/m_\tau = e$: deviation 1.05%. (b) b/s = (lepton ratio) × e: same heuristic origin. (c) $(t/b)/(c/s) = N_c = 3$: physical argument plausible but chirality coupling not computed.

Tier 4b: Claimed but proof has gaps. $\theta_{\mathrm{QCD}} = 0$ from $\pi_3(S^{11}) = \mathbb{Z}_2$: the homotopy group is correct, but the claim that the cascade's topological sectors are classified by $\pi_3(S^{11})$ rather than $\pi_3(\mathrm{SU}(3))$ requires showing how the vector-field realisation of SU(3) on S^{11} modifies the topological sector classification.

What would change my mind. The framework would be significantly weakened by: (i) discovery of a QCD axion; (ii) $w \neq -1$ at $> 5\sigma$ from BAO alone with r_d independently constrained at > 144 Mpc; (iii) discovery of a fourth fermion generation; (iv) a demonstration that the topological obstruction factor is not $2\sqrt{\pi}$ (now derived in Theorem 2.2 and equivalently in Corollary 2.3; falsification would require either invalidating the chirality/quarter-turn argument or invalidating the cascade-primitive identity $2\sqrt{\pi} = N(0) \cdot \Gamma(\frac{1}{2})$); or (v) any absolute mass prediction deviating by $> 5\%$ after subleading corrections are included. Conversely, the framework would be strongly supported by: (vi) derivation of the screening 2π per generation in the α_{em} conjecture (Open Question 3) from the cascade action, which would close $1/\alpha_{\mathrm{em}}$ at 0.006%; (vii) derivation of the colour factor e from first principles; or (viii) any new precision prediction matching observation before the data is checked.

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