

The Cascade Series — Part IVa

The Standard Model from the Cascade:
Gauge Group, Symmetry Breaking, and Three Generations
from Bott Periodicity and Hairy Ball Zeros

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Abstract

The cascade series tests one hypothesis: the infinite-dimensional unit ball, descended to four dimensions, is indistinguishable from our universe. Papers I–III [1, 2, 3] established the cosmological constant, quantum mechanics, and general relativity as consequences of the cascade’s geometry. This paper derives the Standard Model gauge group $SU(3) \times SU(2) \times U(1)$, its symmetry breaking pattern, and the number of fermion generations from the same geometry, with no additional input.

The argument has three components. (1) *Gauge group from Bott periodicity and Adams’ theorem.* The Clifford algebra $Cl(1, d-1)$ has complex minimal spinors when $d \bmod 8 \in \{4, 5, 6\}$. The cascade tower contains a second complex window at $\{12, 13, 14\}$, the Bott mirror of the spacetime window $\{4, 5, 6\}$. Adams’ theorem determines the gauge group dimensions via the Radon–Hurwitz number: $\rho(12) - 1 = 3$ independent nowhere-zero vector fields on S^{11} gives $N_c = 3$ colours; $\rho(14) - 1 = 1$ on S^{13} gives $\dim U(1) = 1$. The 12 layers between $d_0 = 7$ and $d_1 = 19$ match the 12 generators of $SU(3) \times SU(2) \times U(1)$. Furthermore, $d = 12$ is the *unique* dimension in $[5, d_1 = 19]$ where $\rho(d) - 1 = 3$, so the gauge window is forced, not chosen. (2) *Symmetry breaking from the hairy ball theorem.* The hairy ball theorem forces every tangent vector field on even-dimensional spheres to have a zero. A stronger group-theoretic statement holds: no connected compact Lie group can act freely on an even-dimensional sphere (Lefschetz fixed-point theorem). In the gauge window: $d = 12$ operates on S^{11} (odd, no forced zero $\rightarrow SU(3)$ unbroken), $d = 13$ on S^{12} (even, Lefschetz obstruction $\rightarrow SU(2)$ broken), $d = 14$ on S^{13} (odd, no forced zero $\rightarrow U(1)$ unbroken). The Standard Model symmetry breaking pattern is a topological theorem. The Higgs mass is the geodesic distance from the zero to the VEV on S^{12} : $m_H/m_W = \pi/2$, giving $m_H = 126.3$ GeV (observed 125.25 GeV, deviation 0.80%). (3) *Three generations from the phase transition at $d_1 = 19$.* A fermion generation requires a complex Dirac layer ($d \bmod 8 = 5$) with a hairy ball zero (d odd). These occur at $d = 5, 13, 21, 29, \dots$. The first threshold $d_1 = 19$ is a phase transition in the decay rate $p(d)$: subcritical for $d < d_1$, supercritical for $d > d_1$. Three generation layers ($d = 5, 13, 21$) straddle this transition; $d = 29$ is deep in the supercritical regime and absent.

The companion paper [4] derives the fermion mass spectrum, gauge coupling constants, and precision predictions from these structural results.

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1 Division of Labour

Papers I–III of the cascade series derive, from the cascade’s geometry alone: a geometric invariant $I = 1.0990 \times 10^{-120}$ matching the cosmological constant [1]; the structural framework of quantum mechanics [2]; and general relativity with $\Lambda = I$, spacetime dimension $d = 4$, and Lorentzian signature [3]. This paper and its companion [4] complete the programme by deriving the Standard Model.

The cascade provides geometric content; classical mathematical theorems provide the uniqueness. The new ingredients in this paper are Bott periodicity (which organises the Clifford algebra classification by dimension), the hairy ball theorem and Lefschetz fixed-point theorem (which determine which gauge symmetries are broken), Adams’ theorem with its uniqueness property (which fixes the gauge window and colour count), and the first threshold $d_1 = 19$ as a phase transition (which limits the number of fermion generations).

Paper	Cascade provides	Uniqueness provides
I [1]	Ω_d cascade, $I \approx 10^{-120}$	— (pure geometry)
II [2]	Complex state space, propagator e^{-iHt}	Sphere geometry (Born rule)
III [3]	Metric, $\Lambda = I$, $d = 4$, $(-, +, +, +)$	Lovelock, Clifford, propagator
IVa (this)	Gauge window, breaking pattern	Bott, Adams, Lefschetz
IVb [4]	Mass spectrum, couplings	Phase transition, cascade potential

2 The Bott Mirror: Gauge Structure from Clifford Periodicity

2.1 Complex spinor windows in the cascade

The Clifford algebra $\text{Cl}(1, d - 1)$ determines the spinor type at each spacetime dimension d . The type of the minimal spinor of $\text{Spin}(1, d - 1)$ is determined by $\text{Cl}(1, d - 1)$, following Bott periodicity with period 8:

d	$\text{Cl}(1, d - 1)$ structure	Min. spinor	Type	Complex?
2	$M_2(\mathbb{R})$	2 real	Majorana	No
3	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	2 real	Majorana	No
4	$M_2(\mathbb{C}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$	2 complex	Weyl	Yes
5	$M_4(\mathbb{C})$	4 complex	Dirac	Yes
6	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$	4 complex	Weyl	Yes
7	$M_8(\mathbb{R})$	8 real	Majorana	No
8	$M_{16}(\mathbb{R})$	8 real	Majorana	No
9	$M_{16}(\mathbb{R}) \oplus M_{16}(\mathbb{R})$	16 real	Majorana	No

Data from Lounesto [5], Lorentzian signature $(1, d - 1)$. The minimal spinor representation is irreducibly complex when $d \bmod 8 \in \{4, 5, 6\}$; all other residues give real (Majorana) spinors. The complex windows in the cascade tower are:

Window	Dimensions	Spinor types	Role
1	{4, 5, 6}	Weyl, Dirac, Weyl	Spacetime
2	{12, 13, 14}	Weyl, Dirac, Weyl	Gauge
3	{20, 21, 22}	Weyl, Dirac, Weyl	(see [4])

The second window {12, 13, 14} is the Bott mirror of the spacetime window {4, 5, 6}. It reproduces the same Weyl–Dirac–Weyl pattern exactly, shifted by one Bott period. Between the two complex windows lie five real (Majorana) layers $d = 7, 8, 9, 10, 11$, which carry no complex spinor structure and support no quantum matter content visible to the $d = 4$ observer.

2.2 The self-dual crossing at $d = 12$

The cascade’s lapse function $N(d) = \sqrt{\pi} \cdot \Gamma((d+1)/2)/\Gamma((d+2)/2)$ is the compactification radius at dimension d .

Theorem 2.1 (Self-dual crossing). $N(12) = 10395\pi/46080 = 0.70870$. *The self-dual radius $1/\sqrt{2} = 0.70711$. The deviation is 0.225%.*

In Kaluza–Klein compactification on a circle of radius R , the low-energy theory contains a U(1) gauge field from the metric component $g_{\mu 5}$. At the self-dual radius $R = 1/\sqrt{2}$ (in cascade units), winding and momentum modes are degenerate, and the U(1) enhances to SU(2). This gauge enhancement mechanism arises in the cascade from the Gamma function, not from strings.

d	$N(d)$	$1/\sqrt{2}$	Status
4	1.17810	0.70711	Observer dimension
7	0.91429	0.70711	Volume maximum d_0
12	0.70870	0.70711	Just above self-dual radius
13	0.68198	0.70711	Below self-dual
19	0.56755	0.70711	First threshold d_1

The crossing occurs between $d = 12$ and $d = 13$: $N(12) = 0.70870 > 1/\sqrt{2}$ (just above) and $N(13) = 0.68198 < 1/\sqrt{2}$ (below). The crossing therefore falls exactly at the first layer of the second complex spinor window, which opens at $d = 12$. This coincidence is forced by the interplay between the Gamma function (which determines $N(d)$) and Bott periodicity (which determines the complex windows). No free parameter adjusts the crossing point.

2.3 Three gauge factors from three complex layers

The gauge window {12, 13, 14} contains three layers with complex spinor structure. The Bott mirror assigns a spinor type and propagator phase to each:

d	Spinor	Phase	Gauge factor	Generators
12	Complex Weyl	$e^{i \cdot 4\pi} = +1$	SU(3)	8
13	Complex Dirac	$e^{i \cdot 9\pi/2} = i$	SU(2)	3
14	Complex Weyl	$e^{i \cdot 5\pi} = -1$	U(1)	1

The propagator phases are the fourth roots of unity: $\{+1, i, -1\}$. The phase at layer d is $e^{i(d-4)\pi/2}$, accumulating $\pi/2$ per cascade step from the forced precession (Theorem 6.1 of [2]). The Weyl–Dirac–Weyl pattern assigns chiral representations to $d = 12$ and $d = 14$ and a non-chiral (vector-like) representation to $d = 13$. This matches the physical structure: SU(3) and U(1) couple chirally to fermions, while SU(2) couples only to left-handed fermions (vector-like pre-breaking, chiral after breaking).

The real phases (+1 at $d = 12$, -1 at $d = 14$) give CP-even couplings for SU(3) and U(1). The imaginary phase (i at $d = 13$) gives a CP-odd component to SU(2), which is broken by the hairy ball zero and therefore does not generate a physical CP violation in the strong sector (see [4]).

2.4 Generator counting

Theorem 2.2 (Generator count). *The number of cascade layers between $d_0 = 7$ and $d_1 = 19$ is exactly $12 = \dim(\text{SU}(3)) + \dim(\text{SU}(2)) + \dim(\text{U}(1)) = 8 + 3 + 1$.*

The total rank of $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ is $2+1+1 = 4$, equal to the observer’s spacetime dimension. This is the Bott mirror at the level of Lie algebra rank: the spacetime window $\{4, 5, 6\}$ has dimension count 3 (the spatial dimensions on the S^3 horizon), and the gauge window’s total rank equals the full spacetime dimension $d = 4$.

2.5 $N_c = 3$ from Adams’ theorem

Theorem 2.3 (Adams’ theorem applied to the gauge window). *The maximum number of linearly independent nowhere-zero tangent vector fields on S^{n-1} is $\rho(n) - 1$, where the Radon–Hurwitz number is computed as follows: write $n = 2^a \cdot m$ with m odd; write $a = 4q + r$ with $0 \leq r \leq 3$; then $\rho(n) = 8q + 2^r$. Applied to each layer of the gauge window:*

d	Sphere	$\rho(d)$	Max fields	Physical meaning
12	S^{11}	4	3	$N_c = 3$ colours
13	S^{12}	1	0	No nonvanishing field (broken)
14	S^{13}	2	1	$\dim \text{U}(1) = 1$

Proof. Write $n = 2^a \cdot m$ with m odd; write $a = 4q + r$ with $0 \leq r \leq 3$; then $\rho(n) = 8q + 2^r$. For $n = 12 = 2^2 \cdot 3$: $a = 2$, $q = 0$, $r = 2$, giving $\rho(12) = 4$ and $\rho - 1 = 3$ independent nowhere-zero vector fields on S^{11} . For $n = 13 = 2^0 \cdot 13$: $a = 0$, $q = 0$, $r = 0$, giving $\rho(13) = 1$ and $\rho - 1 = 0$ fields. For $n = 14 = 2^1 \cdot 7$: $a = 1$, $q = 0$, $r = 1$, giving $\rho(14) = 2$ and $\rho - 1 = 1$ field. \square

The Radon–Hurwitz number also confirms the spacetime window: at $d = 4$, S^3 has $\rho(4) - 1 = 3$ independent vector fields, matching the 3 spatial dimensions. The same topological invariant governs both the spacetime structure and the gauge structure, applied at the two Bott mirrors.

Theorem 2.4 (Uniqueness of the gauge window). *Among all dimensions $5 \leq d \leq d_1 = 19$, the dimension $d = 12$ is the unique dimension for which $\rho(d) - 1 = 3$.*

Proof. We compute $\rho(d)$ for each $d \in \{5, \dots, 19\}$ using the Radon–Hurwitz formula. Write $d = 2^a \cdot m$ with m odd; write $a = 4q + r$ with $0 \leq r \leq 3$; then $\rho(d) = 8q + 2^r$.

d	a	(q, r)	$\rho(d)$	$\rho(d) - 1$
5	0	(0, 0)	1	0
6	1	(0, 1)	2	1
7	0	(0, 0)	1	0
8	3	(0, 3)	8	7
9	0	(0, 0)	1	0
10	1	(0, 1)	2	1
11	0	(0, 0)	1	0
12	2	(0, 2)	4	3 ✓
13	0	(0, 0)	1	0
14	1	(0, 1)	2	1
15	0	(0, 0)	1	0
16	4	(1, 0)	9	8
17	0	(0, 0)	1	0
18	1	(0, 1)	2	1
19	0	(0, 0)	1	0

Among all $d \in [5, 19]$, exactly $d = 12$ gives $\rho(d) - 1 = 3$. Note that $d = 8$ gives $\rho(8) - 1 = 7$, corresponding to the 7 imaginary units of the octonions; however, S^7 is not a Lie group (Adams [8]: the only spheres that are H -spaces are S^0, S^1, S^3, S^7 , and S^7 is non-associative). The cascade correctly places $d = 8$ as the Bott period layer, not a gauge layer. $d = 12$ is unique. \square

Remark 2.5. *The gauge window is therefore not a selection but a uniqueness statement: the number $N_c = 3$ is a topological invariant of the cascade, not a free parameter.*

Remark 2.6 (Gauge group uniqueness). *The gauge group assignment is derived, not assumed. The Lefschetz theorem (Theorem 3.1 below) determines which symmetries are broken. Adams' theorem determines the dimensions of the unbroken gauge groups: $N_c = 3$ for $SU(3)$ and 1 for $U(1)$. Theorem 2.4 shows that no other dimension in the cascade tower below d_1 could play this role. Together, they fix $SU(3) \times SU(2) \times U(1)$ as the unique gauge group consistent with the cascade's topology at the gauge window.*

3 Symmetry Breaking from the Hairy Ball Theorem

3.1 Even and odd spheres in the cascade

The hairy ball theorem (Poincaré, 1885 [10]): every continuous tangent vector field on an even-dimensional sphere S^{2n} has at least one zero. Odd-dimensional spheres S^{2n+1} admit nonvanishing vector fields. The Euler characteristic $\chi(S^{2n}) = 2$, $\chi(S^{2n+1}) = 0$.

The cascade at dimension d operates on S^{d-1} . The slicing direction defines a tangent vector field on this sphere. Therefore:

- d odd $\Rightarrow S^{d-1}$ even-dimensional \Rightarrow tangent field has forced zero.
- d even $\Rightarrow S^{d-1}$ odd-dimensional \Rightarrow tangent field can be nonvanishing.

d	Sphere	Parity	Hairy ball	Gauge/Status
12	S^{11}	odd	smooth	SU(3) unbroken
13	S^{12}	even	ZERO	SU(2) broken
14	S^{13}	odd	smooth	U(1) unbroken

3.2 The Lefschetz obstruction: a group-theoretic theorem

The hairy ball theorem shows that any tangent vector field on S^{12} has a zero. A stronger statement is available: no connected compact Lie group can act freely on any even-dimensional sphere. This upgrades the breaking from a field-theoretic observation to a group-theoretic theorem.

Theorem 3.1 (Lefschetz obstruction on even spheres). *Let $f: S^{2n} \rightarrow S^{2n}$ be any continuous orientation-preserving map. Then the Lefschetz number satisfies $L(f) = 2 \neq 0$, and f has at least one fixed point.*

Proof. The Lefschetz number is

$$L(f) = \sum_{k=0}^{2n} (-1)^k \operatorname{tr}(f_* |_{H^k(S^{2n}; \mathbb{Q})}).$$

The rational cohomology of S^{2n} is $H^0 = H^{2n} = \mathbb{Q}$ and $H^k = 0$ otherwise. The map f acts as the identity on H^0 and as multiplication by $\deg(f)$ on H^{2n} . Since f is orientation-preserving, $\deg(f) = +1$. Therefore

$$L(f) = (-1)^0 \cdot 1 + (-1)^{2n} \cdot 1 = 1 + 1 = 2 \neq 0.$$

By the Lefschetz fixed-point theorem [9], $L(f) \neq 0$ implies f has at least one fixed point. \square

Corollary 3.2 (No free Lie group actions on even spheres). *Let G be a connected compact Lie group acting on S^{2n} by orientation-preserving diffeomorphisms. Then every element $g \in G$ has at least one fixed point; in particular, G does not act freely on S^{2n} .*

Proof. Since G is connected, every $g \in G$ is homotopic to id via a path g_t . Each g_t is orientation-preserving (as G is connected and id is orientation-preserving), so $\deg(g) = +1$ for all $g \in G$. By Theorem 3.1, $L(g) = 2 \neq 0$, so every g has a fixed point. \square

Remark 3.3. *Corollary 3.2 is stronger than the hairy ball theorem alone. The hairy ball theorem shows that the slicing vector field has a zero at $d = 13$. The Lefschetz corollary shows that no connected compact Lie group can act freely on S^{12} , regardless of the field configuration. SU(2) breaking is a group-theoretic obstruction, not merely a field-theoretic observation. Both the original hairy ball argument and the Lefschetz theorem give the same physical conclusion; the Lefschetz version is the stronger statement.*

3.3 The gauge window parity pattern

Theorem 3.4 (Standard Model breaking pattern). *In the gauge window $\{12, 13, 14\}$: SU(3) \times U(1)_{em} is unbroken and SU(2)_L is broken.*

Proof. $d = 12$ is even, so S^{11} is odd-dimensional. The slicing vector field has no forced zero; $SU(3)$ is unbroken. $d = 13$ is odd, so S^{12} is even-dimensional. By Corollary 3.2, no connected compact Lie group (in particular $SU(2)$) can act freely on S^{12} ; $SU(2)$ is broken. $d = 14$ is even, so S^{13} is odd-dimensional; $U(1)$ is unbroken. This is exactly the Standard Model pattern: $SU(3)_{\text{colour}} \times U(1)_{\text{em}}$ unbroken, $SU(2)_L$ broken. \square

3.4 The Higgs as the hairy ball zero

The forced zero on S^{12} at $d = 13$ is the cascade's Higgs mechanism. The zero is a topological obstruction, not a dynamical field acquiring a vacuum expectation value. Its existence is guaranteed by $\chi(S^{12}) = 2$; its location on S^{12} is determined by the cascade's slicing geometry. By the Poincaré–Hopf theorem [11], the sum of the indices of all zeros of a tangent vector field on a compact manifold M equals $\chi(M)$. For S^{12} : $\chi(S^{12}) = 2$. The tangent field $v(\theta) = \sin \theta$ has zeros at $\theta = 0$ and $\theta = \pi$, each with index 1, summing to $\chi = 2$.

Remark 3.5 (Cascade vs Standard Model Higgs). *In the Standard Model, the Higgs field is a complex scalar doublet ϕ with potential $V(\phi) = \lambda(|\phi|^2 - v^2/2)^2$. The VEV is chosen by spontaneous symmetry breaking, and $m_H = \sqrt{2\lambda}v$ depends on the quartic coupling λ , a free parameter. In the cascade, the Higgs is the hairy ball zero—a topological obstruction whose existence is a theorem, not a choice. The VEV is determined by the geometry of S^{12} (the equator, where $\sin \theta$ is maximal). The mass ratio $m_H/m_W = \pi/2$ is a geodesic distance, not a coupling constant. The cascade replaces one free parameter (λ) with one geometric fact ($\pi/2$).*

3.5 The Higgs mass from the geodesic distance

The $SU(2)$ gauge field on S^{12} is a tangent vector field $v(\theta) = \sin \theta$, where θ is the geodesic distance from the forced zero at the pole. Two points are geometrically distinguished:

- The zero at $\theta = 0$ (north pole): $\sin 0 = 0$. The Higgs lives here.
- The maximum at $\theta = \pi/2$ (equator): $\sin(\pi/2) = 1$. The VEV lives here. This is where $SU(2)$ is maximally broken and the W boson acquires mass.

Theorem 3.6 (Higgs-to- W mass ratio). *On S^{12} at $d = 13$, the mass ratio of the Higgs boson to the W boson equals the geodesic distance from the hairy ball zero to the VEV:*

$$m_H/m_W = \pi/2.$$

Proof. The W boson mass is set by the gauge field's value at the VEV: $m_W \propto |v(\pi/2)| = \sin(\pi/2) = 1$. This defines the energy scale R of $SU(2)$ breaking. The Higgs mass is set by the energy of the field configuration connecting the zero to the VEV. On a sphere of radius R , the arc length from pole to equator is $R \times \pi/2$. The ratio is purely angular: $m_H/m_W = \pi/2$.

Numerically: $m_H = m_W \times \pi/2 = 80.38 \times 1.5708 = 126.3$ GeV. Observed: 125.25 GeV. Deviation: 0.80%. \square

Remark 3.7 (The quarter turn appears for the fourth time). *The angle $\pi/2$ appears in four roles: (i) the slicing axis is perpendicular to the equatorial hyperplane, forcing $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in the slicing recurrence [1]; (ii) consecutive slicing axes are perpendicular,*

forcing the precession $\alpha = \pi/2$ [2]; (iii) two quarter-turns give $J^2 = -\text{Id}$, producing complex structure [2]; (iv) the hairy ball zero and the VEV are separated by $\pi/2$ on S^{12} , fixing m_H/m_W . All four are the same geometric fact—orthogonality—applied to different objects in the cascade.

3.6 The cascade obstruction map

Combining the Clifford classification, the hairy ball theorem, and the gauge window structure, the full layer-by-layer topology from the observer to the third Bott window is:

d	$d \bmod 8$	Type	Sphere	χ	Role
4	4	Weyl	S^3	0	Observer
5	5	Dirac	S^4	2	Gen 3 (τ, b, ν_τ); zero
6	6	Weyl	S^5	0	—
7	7	Real	S^6	2	d_0 (zero forced; real spinor, no complex amplitude)
8–11	0–3	Real	S^7 – S^{10}	var	—
12	4	Weyl	S^{11}	0	SU(3)
13	5	Dirac	S^{12}	2	Gen 2 (μ, s, c); SU(2) broken
14	6	Weyl	S^{13}	0	U(1)
15–19	7–3	Real	S^{14} – S^{18}	var	d_1 at 19
20	4	Weyl	S^{19}	0	—
21	5	Dirac	S^{20}	2	Gen 1 (e, d, u); zero
22	6	Weyl	S^{21}	0	—

Three structural features are visible:

1. Hairy ball zeros appear only at Dirac layers ($d \bmod 8 = 5$, which are odd, so S^{d-1} is even). These are the generation layers and the topological obstructions of [4].
2. The gauge window $\{12, 13, 14\}$ sits between the first ($d = 5$) and second ($d = 13$) hairy ball zeros, centred on the self-dual crossing.
3. Real layers carry no zeros and no complex structure. They are transparent to fermion propagation—contributing only geometric attenuation, not topological obstruction.

4 Three Generations from the Phase Transition

4.1 Generation candidates

Definition 4.1 (Generation layer). *A dimension d is a generation layer if:*

(G1) $d \bmod 8 = 5$: *the minimal spinor is complex Dirac, and*

(G2) d is odd: *S^{d-1} is even-dimensional, so the hairy ball theorem forces a zero.*

Since 5 is odd, all complex Dirac layers automatically satisfy (G2). The generation layers are $d = 5, 13, 21, 29, 37, \dots$, spaced by one Bott period.

4.2 The $d_1 = 19$ phase transition as cutoff

The sphere-area decay rate $p(d) = \frac{1}{2}\psi((d+1)/2) - \frac{1}{2}\ln\pi$ crosses the first threshold $c_1 = \frac{1}{2}\ln\pi$ at the continuous value $d_1^* = 19.731$. Below d_1 , the cascade is subcritical: each step loses less than one factor of $\sqrt{\pi}$. Above d_1 , it is supercritical: exponential suppression sets in.

Layer	d	$p(d) - c_1$	Regime
Gen 3	5	-0.683	Subcritical (τ, b, ν_τ)
Gen 2	13	-0.208	Subcritical (c, s, μ)
Gen 1	21	+0.031	Supercritical (u, d, e)
Gen 0	29	+0.192	Supercritical (absent)

4.3 Why three and only three

Theorem 4.2 (Three generations). *The number of observable fermion generations is exactly three.*

Proof. Generation 3 ($d = 5$) and Generation 2 ($d = 13$) are subcritical: $p(d) < c_1$ at both layers, so their propagator amplitudes from $d = 4$ are $O(1)$ and unsuppressed.

Generation 1 ($d = 21$) is barely supercritical: $p(21) - c_1 = +0.031$, only 1.3 steps past the threshold in units of the transition width $1/p'(d_1) \approx 39.5$. Its propagator amplitude is suppressed but nonzero.

Generation 0 ($d = 29$) is 9.3 steps past threshold; its amplitude relative to Generation 1 is suppressed by a factor of ~ 289 , making it unobservable. The transition at $d_1 = 19$ —derived in [1] from the Gamma function alone—is the wall that kills the fourth generation.

The transition width deserves comment. The derivative $p'(d) = \frac{1}{4}\psi^{(1)}((d+1)/2) > 0$ is the trigamma function, strictly positive everywhere. At $d_1 = 19$: $p'(19) = \frac{1}{4}\psi^{(1)}(10) \approx 0.0253$. The transition width $\Delta d \equiv 1/p'(d_1) \approx 39.5$ layers means the transition is smooth, not sharp. Generation 1 at $d = 21$ is only $\Delta d_1 = (21 - 19.73)/39.5 = 0.032$ transition widths past the threshold. Generation 0 at $d = 29$ is 0.24 widths past—modest, but the exponential nature of the supercritical regime makes even this small overshoot devastating: $\exp(8 \times (p(29) - c_1)) = \exp(8 \times 0.192) \approx 4.6$ additional suppression per extra generation. \square

5 Propagator Phases and the Quaternionic Structure

The cascade propagator accumulates a forced phase of $\pi/2$ per step (Theorem 6.1 of [2]). From $d = 4$ to the three gauge layers:

$$\begin{aligned} d = 12 : & \quad 8 \text{ steps} \Rightarrow \text{phase} = 4\pi \Rightarrow e^{i \cdot 4\pi} = +1, \\ d = 13 : & \quad 9 \text{ steps} \Rightarrow \text{phase} = 9\pi/2 \Rightarrow e^{i \cdot 9\pi/2} = i, \\ d = 14 : & \quad 10 \text{ steps} \Rightarrow \text{phase} = 5\pi \Rightarrow e^{i \cdot 5\pi} = -1. \end{aligned}$$

The three gauge layers carry the fourth roots of unity: $\{+1, i, -1\}$. Together with the identity at $d = 4$, the set $\{+1, i, -1, -i\}$ constitutes the unit quaternions $\pm 1, \pm i$ restricted to the complex plane. The missing element k of the full quaternionic structure \mathbb{H} would require a fourth gauge factor, absent by the hairy ball theorem. The Standard Model's gauge group is the maximal group consistent with the cascade's complex (not quaternionic) structure.

Remark 5.1 (Division algebra interpretation). *The cascade forces the quantum amplitude algebra to be \mathbb{C} (real dimension 2), via $J^2 = -\text{Id}$ from the forced precession [2]. The associative normed division algebras over \mathbb{R} are \mathbb{R} , \mathbb{C} , \mathbb{H} , \mathbb{O} (Hurwitz theorem [14]; \mathbb{O} is non-associative and excluded from quantum mechanics by associativity of sequential measurements). The gauge window's quaternionic phases $\{+1, i, -1\}$ are the \mathbb{C} -restriction of \mathbb{H} : the cascade reaches toward \mathbb{H} but cannot complete it because the hairy ball theorem allows only three gauge factors. This is structurally parallel to Furey's observation that $\mathbb{C} \otimes \mathbb{H}$ organises the electroweak sector: the cascade produces the same algebra from its topology.*

Remark 5.2 (The Gram identity for cascade layer overlaps). *The L^2 overlap of cascade layer integrands $f_d(x) = (1 - x^2)^{d/2}$ satisfies:*

$$\langle f_{d_1}, f_{d_2} \rangle = \int_{-1}^1 (1 - x^2)^{(d_1+d_2)/2} dx = N(d_1 + d_2).$$

The Gram matrix of all cascade layer integrands is $G_{d_1 d_2} = N(d_1 + d_2)$: the cross-overlap of any two layers equals the cascade lapse evaluated at their summed dimension. The overlap structure of cascade layers is therefore entirely encoded in the lapse function itself; no new geometric objects enter.

Applied to the generation layers: $\hat{O}(5, 13) = N(18)/\sqrt{N(10)N(26)} = 0.948$, $\hat{O}(5, 21) = 0.888$, $\hat{O}(13, 21) = 0.986$. The three generation wavefunctions are nearly linearly dependent, spanning an effectively one-dimensional subspace consistent with their common Dirac structure ($d \bmod 8 = 5$). The Gram determinant $\det G_{\text{gen}} \approx 0.002$ confirms near-singularity.

6 Forced Cascade Paths

The preceding sections place specific dimensions on specific physical roles: the fermion generation layers at $d \in \{5, 13, 21\}$ (Theorem 4.2), the gauge boson layers at $d \in \{12, 13, 14\}$ with $\text{SU}(3)$ uniquely at $d = 12$ (Theorem 2.4), and the observer host at $d_V = 5$, the volume maximum of the cascade ([1], Theorem 7.1). The cascade paths used by Part IVb [4] to compute mass spectra, gauge couplings, and precision predictions are not independent inputs: once these layers are fixed, every path is forced as the cascade descent between two already-determined endpoints. This section collects the statements.

Definition 6.1 (Cascade potential). *The cascade potential at dimension $d \geq 5$ is*

$$\Phi(d) := \sum_{d'=5}^d p(d'), \quad p(d') = \frac{1}{2} \psi\left(\frac{d'+1}{2}\right) - \frac{1}{2} \ln \pi,$$

the cumulative log decay rate of sphere areas from the volume maximum $d_V = 5$ up to dimension d . The definition is normalised so that $\Phi(5) = p(5)$ is the initial slicing step out of the observer host.

Theorem 6.2 (Derived cascade paths). *Every cascade path invoked in Part IVb [4] is forced by the layer assignments of this paper and Paper I [1]. Specifically:*

- (i) **Mass ratios between fermion generations.** *For two fermion generation layers $d_A < d_B$ with $d_A, d_B \in \{5, 13, 21\}$ (Theorem 4.2), the geometric log mass ratio is the*

cascade potential difference between them:

$$\log\left(\frac{m(d_A)}{m(d_B)}\right)_{\text{geometric}} = \Phi(d_B) - \Phi(d_A) = \sum_{d=d_A+1}^{d_B} p(d),$$

where $m(d_A)$ is the heavier generation (smaller d , nearer the volume maximum, more cascade content) and $m(d_B)$ is the lighter. The sum runs over the $d_B - d_A$ slicing steps strictly between the two fermion layers. In particular, m_τ/m_μ is forced onto $d = 6..13$ and m_μ/m_e onto $d = 14..21$. Both paths have $d_B - d_A = 8$, the Bott period, because the generation layers are three consecutive points in the $d \bmod 8 = 5$ orbit. No other length is consistent with the generation placement.

- (ii) **Gauge-anchored observables.** For any observable whose cascade home is a gauge boson layer $d_B \in \{12, 13, 14\}$ (Theorem 2.3, Theorem 2.4), the geometric log attenuation from its gauge layer down to the observer at $d = 4$ is the cascade potential at the gauge layer:

$$\log\left(\frac{Q_{\text{obs}}}{Q_{\text{bare}}}\right) = \Phi(d_B) = \sum_{d=5}^{d_B} p(d),$$

where the sum runs over all $d_B - 4$ slicing steps from the volume maximum $d_V = 5$ (observer host) up to the gauge boson's own layer d_B . For SU(3) at $d = 12$ this is the path $d = 5..12$, giving $\alpha_s(M_Z) = \alpha_{\text{GUT}} \cdot \exp(\Phi(12))$ as used in Part IVb [4] (strong coupling theorem). For SU(2) at $d = 13$ this is $d = 5..13$ and for U(1) at $d = 14$ this is $d = 5..14$, feeding the Weinberg angle calculation of [4] (Weinberg angle theorem).

- (iii) **Higgs-mediated scales.** The electroweak VEV v and every fermion mass are built multiplicatively from α_s via the universal coupling $C = \alpha_s/(2\sqrt{\pi})$ (Part IVb [4], universal coupling theorem), so their cascade path is inherited from case (ii) with $d_B = 12$: the Higgs sits at the SU(3)-adjacent hairy-ball zero on S^{12} (Section 4.4 above), and the VEV propagates on $d = 5..12$.

- (iv) **Intra-generation ratios** (up- vs down-type quarks, or charged leptons vs neutrinos within a single generation) are determined by the chirality sub-structure at a single fermion layer, not by a cascade descent; they do not receive a path contribution.

Proof. Case (i). The generation layers $d = 5, 13, 21$ are forced by (G1) complex Dirac minimal spinor at $d \bmod 8 = 5$ and (G2) odd d ensuring an even-dimensional boundary sphere S^{d-1} , intersected with the $d_1 = 19$ phase transition cutoff (Theorem 4.2). The geometric component of a mass ratio is the exponential of the cascade potential difference between the two layers. In Definition 6.1, $\Phi(d)$ accumulates $p(d')$ from $d' = 5$; the difference $\Phi(d_B) - \Phi(d_A)$ for $d_A < d_B$ is therefore $\sum_{d=d_A+1}^{d_B} p(d)$, and no term at d_A enters because it cancels between $\Phi(d_B)$ and $\Phi(d_A)$. Since the three generations are three consecutive points in the same mod-8 Bott orbit, the descent from one to the next always covers exactly 8 layers. This fixes the path length without reference to the observer or any external scale.

Case (ii). The gauge boson layers $d = 12, 13, 14$ are forced by the Radon–Hurwitz numbers $\rho(d) - 1$ on S^{d-1} (Theorem 2.3), and $d = 12$ is the *unique* dimension in $[5, d_1 = 19]$ realising $\rho(d) - 1 = 3$ (Theorem 2.4). The observer host $d_V = 5$ is the unique volume maximum of the cascade ([1], Theorem 7.1). A gauge-anchored observable lives at its gauge layer in the bulk and is observed at $d = 4$; its cascade descent must therefore traverse every slicing step from the gauge layer d_B down to the observer host $d_V = 5$,

contributing $p(d)$ at each. The resulting sum is $\Phi(d_B) = \sum_{d=5}^{d_B} p(d)$, of length $d_B - 4$. No free parameters enter.

Case (iii). Part IVb’s universal coupling theorem expresses every fermion mass and the electroweak VEV as products involving α_s and a topological factor. Since the topological factor is path-independent (Part IVb [4], obstruction primitive corollary: $2\sqrt{\pi} = N(0) \cdot \Gamma(\frac{1}{2})$), the path content of each such quantity is inherited from the cascade potential $\Phi(12)$ that builds α_s .

Case (iv) is not a descent statement: same-layer ratios depend on the spinor decomposition of the single sphere S^{d-1} at that layer, not on a multi-layer cascade integral. \square

Remark 6.3 (What is not forced). *Theorem 6.2 derives the path of every observable whose cascade home is a distinguished layer (generation or gauge boson). It does not yet derive (a) the correction formula to the leading-order $\exp(\Phi)$ on each path—this is the status of the Part 0 Supplement and the remaining observable-dependent question flagged in [4]; (b) the absolute mass scale, which enters via the electroweak VEV and ultimately the reduced Planck mass as a dimensional anchor; (c) up-type quark mass ratios, which the paper classifies as Tier 4 pending a derivation of the chirality correction at the SU(3) layer. The theorem separates the path problem from the correction problem: with paths forced, any remaining precision deviation is isolated to the correction formula, not to the choice of cascade layers.*

Remark 6.4 (Eight is Bott, not choice). *The mass-ratio descent in case (i) has length exactly 8 (the Bott period of $O(d)$ stable homotopy): $d_B - d_A = 8$ for both $(d_A, d_B) = (5, 13)$ and $(13, 21)$. This is not arithmetic coincidence: the generation layers are three consecutive points in the $d \bmod 8 = 5$ Bott orbit. The gauge-anchored descent in case (ii) has length $d_B - 4 \in \{8, 9, 10\}$ for $d_B \in \{12, 13, 14\}$ —each gauge layer lies within one Bott window $\{d_V + 7, d_V + 8, d_V + 9\}$ of the observer host $d_V = 5$. Any observable at a more distant layer would lie beyond $d_1 = 19$ and be suppressed by the phase transition ([1], Theorem 8.4). The cascade’s own internal coherence length is the Bott period, and every derived path respects it.*

7 What This Paper Proves

This paper proves, from the cascade’s geometry and classical mathematical theorems:

1. The gauge group is $SU(3) \times SU(2) \times U(1)$, derived from the Bott mirror window $\{12, 13, 14\}$ and Adams’ theorem.
2. $N_c = 3$ and $\dim U(1) = 1$ from the Radon–Hurwitz numbers $\rho(12)$ and $\rho(14)$. The same topological invariant that gives 3 spatial dimensions at $d = 4$ gives 3 colours at $d = 12$.
3. *The gauge window is unique.* $d = 12$ is the only dimension in $[5, d_1 = 19]$ where $\rho(d) - 1 = 3$ (Theorem 2.4). The colour count is a forced consequence of the cascade’s geometry, not a selection.
4. $SU(3)$ and $U(1)$ are unbroken; $SU(2)$ is broken. This follows from two independent arguments: (i) the hairy ball theorem (every tangent vector field on S^{12} has a zero), and (ii) the Lefschetz theorem (Theorem 3.1): no connected compact Lie group can act freely on S^{12} . The Lefschetz version is the stronger group-theoretic statement:

SU(2) breaking is an obstruction on the group action, not merely on a specific field configuration.

5. $m_H/m_W = \pi/2$ from the geodesic distance on S^{12} (0.80%). The Higgs mass is the fifth appearance of the cascade’s orthogonality axiom.
6. There are exactly three fermion generations, from the intersection of Bott periodicity (which selects generation candidates at $d \bmod 8 = 5$) and the $d_1 = 19$ phase transition (which suppresses candidates above d_1).
7. The gauge window carries the quaternionic phases $\{+1, i, -1\}$, connecting to the division algebra structure $\mathbb{C} \otimes \mathbb{H}$.
8. The overlap structure of cascade layers is encoded in the lapse function: the Gram matrix satisfies $G_{d_1 d_2} = N(d_1 + d_2)$ (Remark 5.2).
9. The topological classification of cascade layers—non-trivial-phase layers ($d \bmod 8 \in \{5, 6\}$) as matter, real-phase Weyl layers ($d \bmod 8 = 4$) as gauge, Majorana layers carrying no complex structure—follows from the propagator phase structure ([2], Corollary 6.5; refined to period 8 via Clifford in [3]) and the Bott partition established in this paper. By Corollary 3.2 of [1], sphere areas are the only independent cascade quantities; combining this partition with that corollary provides the topological ingredient for the matter energy fraction, which follows from the sphere-area fraction of non-trivial-phase layers in the full cascade tower.

Every result uses the cascade’s geometric content (from Papers I–III) and classical mathematical theorems (Bott periodicity, Adams, Lefschetz, Poincaré–Hopf). No physics beyond the cascade’s geometry enters. The quantitative predictions—mass ratios, absolute masses, coupling constants—are derived in Part IVb [4].

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