The Cascade Series — Part III

General Relativity, Four Dimensions, and Lorentzian Signature from the Cascade: Lovelock Uniqueness, Spinor Compatibility, and the Cosmological Constant

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Abstract

The sphere-area cascade [1] derives a geometric invariant $I = 1.0990 \times 10^{-120}$ from orthogonality alone. The companion paper [2] shows that the cascade geometry produces complex quantum mechanics via a forced precession angle $\alpha = \pi/2$. This paper derives, from the same orthogonality axiom and with no additional empirical input: (1) general relativity with cosmological constant $\Lambda = I$; (2) the observer's spacetime dimension $d = 4$; and (3) Lorentzian metric signature $(-, +, +, +)$.

The argument has four components. **The cascade metric** (Section 4): the slicing recurrence produces a foliated FRW-type metric with a natural lapse function and extrinsic curvature. **Lovelock uniqueness** (Section 3): the Einstein equation with cosmological constant is the unique divergence-free symmetric rank-2 metric equation in exactly four spacetime dimensions. **Dimension derivation** (Section 9): $d = 4$ is the unique dimension satisfying simultaneously (C1) the Clifford algebra classification requires irreducibly complex spinors (forced by the cascade's $J^2 = -\mathrm{Id}$), and (C2) Lovelock uniqueness holds. The intersection of these conditions is $\{4\}$. A third independent characterisation is established in Corollary 9.4: the Lorentzian cascade metric has Ricci scalar $R^{(n)} = (n-1)(n-4)/a^4$, vanishing uniquely at $n = 4$. **Lorentzian signature** (Section 10): the cascade produces both a Euclidean geometry and a quantum propagator $e^{-i\lambda x}$ with $\lambda > 0$ from the forced precession. Given the physical identification hypothesis, the propagator e^{-iHt} with $H > 0$ is oscillatory and unitary: Lorentzian time evolution. The signature $(-, +, +, +)$ follows without semiclassics or analytic continuation.

The cosmological constant $\Lambda = I$ is derived, not fitted. The cascade's dark energy equation of state is $w = -1$ exactly: $\Lambda = I$ is a fixed geometric constant with no time-evolution mechanism, so $w = -1$ by definition. The Gauss–Bonnet correction at $d = 5$ vanishes identically by two independent mechanisms (totally umbilic cancellation and parity of the residual integrand), confirming the prediction is structurally stable. The supplement (Section 14) proves the compactification radius at each step is $R_{\mathrm{eff}}(d) = 1/\sqrt{d+3}$ exactly (from the Beta function), and presents the exact Lorentzian Gauss–Codazzi analysis: the cascade metric contributes $w = 1/3$

(radiation) in cascade time, and $K_{\mu\nu} = 0$ at every equatorial embedding eliminates the extrinsic curvature mechanism. These results resolve the question of the cascade's matter fraction: the geometric route via Gauss–Codazzi is definitively closed, and the matter fraction Ω_m is a topological quantity determined by the Bott partition of cascade layers, not by the cascade metric geometry.

Contents

1 Division of Labour and Summary of the Series

1.1 The architecture

The three-paper core rests on one hypothesis and three uniqueness theorems.

The hypothesis (this paper, Section 2): the cascade's abstract geometry is our physics. Without it, the series is pure mathematics; with it, every physical result is a prediction that tests the hypothesis.

The structure: the cascade provides geometric content; classical uniqueness theorems identify the unique physical theory consistent with that content.

Paper	Cascade provides	Uniqueness provides
I [1]	Ω_d cascade, $I \approx 10^{-120}$	— (pure geometry)
II [2]	Complex state space, propagator e^{-iHt}	Sphere geometry (Born rule)
III (this)	Metric, $\Lambda = I$, $d = 4$, $(-,+,+,+)$	Lovelock, Clifford, propagator

1.2 What this paper proves

Given the physical identification hypothesis (Section 2), this paper derives four results from the orthogonality axiom of [1]:

1. The cascade produces a foliated spacetime geometry with a natural lapse function $N(d) = \sqrt{\pi} \cdot R(d)$ and a FRW-type 4D metric.
2. By Lovelock's theorem, the unique gravitational equation available to a 4D observer is the Einstein equation; the cosmological constant equals the cascade invariant $\Lambda = I$.
3. $d = 4$ is the unique spacetime dimension satisfying both the cascade's forced complex spinor requirement and Lovelock uniqueness.
4. Lorentzian metric signature $(-,+,+,+)$ is the unique signature consistent with the cascade's forced propagator e^{-iHt} with $H > 0$.

2 The Physical Identification Hypothesis

The cascade is a mathematical object: a sequence of sphere areas $\{\Omega_d\}$ derived from orthogonality. Paper I [1] is pure mathematics; it makes no physical claim beyond recording the numerical coincidence $I \approx \rho_\Lambda / M_{\mathrm{Pl}}^4$. Paper II [2] shows the cascade's state space has the structure of quantum mechanics. This paper derives the cascade's gravitational structure. Both Papers II and III require one physical input.

Definition 2.1 (Physical identification hypothesis)**.** *The infinite-dimensional unit ball, descended to four dimensions, is indistinguishable from our universe. Concretely: the cascade metric is the spacetime metric; the cascade propagator is the physical time-evolution operator; the cascade invariant I is the cosmological constant. This hypothesis is the claim that the cascade's abstract geometry is our physics. Without it, the series is pure mathematics; with it, every physical result is a prediction that tests the hypothesis.*

Remark (Uniformity of the physical identification)**.** *Definition 2.1 maps cascade content to physical energy uniformly across all layer types. By Corollary 3.2 of [1], sphere area* Ω_{d-1} *is the only independent cascade quantity at level d; every other cascade object is derived from sphere areas. Therefore the identification assigns energy* $C \cdot \Omega_{d-1}$ *at each level with C a universal constant—no layer-type-dependent modulation is available, because no independent geometric quantity exists to carry such modulation. Matter and vacuum are observer labels applied to the Bott partition of cascade layers: the propagator phase classification ([2], Corollary 6.5; refined to period 8 via Clifford in this paper, Section 9) establishes the partition topologically, and the fermion generation structure ([3]) identifies which sector is matter via the hairy ball zeros and observed generations. The cascade geometry itself makes no distinction between the sectors.*

Remark. *The physical identification hypothesis does not assume: the dimension of spacetime (derived as* $d = 4$ *in Section 9); the metric signature (derived in Section 10); the form of the gravitational equation (derived via Lovelock); or the value of* Λ *(derived as I from [1]). It assumes only that the cascade corresponds to something physical. This is the minimal assumption needed to connect abstract geometry to physics.*

3 Lovelock's Theorem and the Privilege of Four Dimensions

Theorem 3.1 (Lovelock [4, 5])**.** *Let* (M, g) *be a smooth d-dimensional Lorentzian manifold with* $d = 4$*. The most general symmetric divergence-free rank-2 tensor constructed from* $g_{\mu\nu}$ *and its first and second derivatives is*

$$E^{\mu\nu} = a\, G^{\mu\nu} + b\, g^{\mu\nu},$$

where $G^{\mu\nu}$ *is the Einstein tensor and a, b are constants. Setting* $a = 1$*,* $b = -\Lambda$ *recovers the Einstein equation with cosmological constant.*

The theorem is dimension-dependent. In d spacetime dimensions, the most general such tensor is a sum of $\lfloor d/2 \rfloor$ independent Lovelock terms plus Λ.

Dimension d	Lovelock terms	Gravity unique?
3	Λ only	No propagating gravity
4	$G^{\mu\nu} + \Lambda g^{\mu\nu}$ only	Yes: GR is unique
5, 6	Einstein + Gauss–Bonnet + Λ	No: free coupling
≥ 7	Further Lovelock terms + Λ	No

Remark. *Lovelock uniqueness at* $d = 4$ *is a theorem about tensor equations, not a consequence of the cascade. The cascade's role is to provide the metric that Lovelock needs as input. Section 9 derives* $d = 4$ *from the consistency of the cascade's quantum structure with Lovelock uniqueness.*

4 The Cascade Metric

4.1 Foliation from slicing

The cascade's slicing recurrence decomposes B^{d+1} into d-balls parametrised by the perpendicular coordinate $x \in [-1, 1]$. Each cross-section at height x is a d-ball of radius

$r(x) = \sqrt{1-x^2}$. This is a foliation; x plays the role of time (identified in [2], Section 7 via the irreversibility of slicing). The full $(d+1)$-dimensional metric is:

$$ds^2 = dx^2 + (1-x^2)\, d\sigma_{d-1}^2.$$

4.2 The lapse function

The ratio of adjacent volumes is the cascade's lapse:

$$N(d) = \sqrt{\pi} \cdot R(d), \qquad R(d) = \frac{\Gamma((d+1)/2)}{\Gamma((d+2)/2)} \approx \sqrt{\frac{2\pi}{d}} \quad (d \gg 1).$$

At $d = 4$: $N(4) = \sqrt{\pi} \cdot \Gamma(5/2)/\Gamma(3) = 3\pi/8 \approx 1.178$. As $d \to \infty$: $N \to 0$ (no temporal separation at the cascade's geometric origin).

4.3 The extrinsic curvature

The extrinsic curvature of the leaf at height x is $K = -x/\sqrt{1-x^2}$. At the equator $x = 0$: $K = 0$. This is the cascade's Hubble parameter.

4.4 The 4D projected metric

Projecting onto a 4-dimensional subspace, the 4D observer's metric is:

$$ds_{4\mathrm{D}}^2 = -dt^2 + a(t)^2\, d\sigma_3^2.$$

This is the FRW metric with unit lapse. Isotropy follows from the cascade's spherical symmetry; homogeneity from the uniform slicing recurrence. The Lorentzian sign $g_{tt} = -1$ is derived in Section 10.

5 The Cosmological Constant

5.1 From cascade invariant to Λ

By [1], Theorem 9.2:

$$I = \frac{9\,\Omega_{19}\,\Omega_{217}}{\pi^2} = 1.0990 \times 10^{-120}.$$

The 4D observer's vacuum energy satisfies $\rho_\Lambda / M_{\mathrm{Pl}}^4 \approx 1.1 \times 10^{-120}$. The identification $\Lambda = I$ is not a fit: I is fixed by π independently of any physical measurement. The agreement is 0.04% under the reduced-Planck-mass convention.

5.2 Why Λ is not a free parameter

In standard GR, Λ is a free parameter: Lovelock permits any b in $E^{\mu\nu} = aG^{\mu\nu} + bg^{\mu\nu}$. The cascade removes this freedom. The constant $b = -\Lambda$ is fixed by the cascade invariant I; the observer does not choose it. The cosmological constant problem—why $\Lambda \sim 10^{-120} M_{\mathrm{Pl}}^4$—is answered by the hierarchy theorem of [1]: $\log_{10}(1/\Omega_{d_2}) \approx \pi e^{2\sqrt{\pi}}(2\sqrt{\pi} - 1)/\ln 10 \approx 120.3$ orders of magnitude arise from the Gamma function's superexponential growth between $d_1 = 19$ and $d_2 = 217$.

6 The Dark Energy Equation of State

Theorem 6.1 (Dark energy equation of state)**.** *The cascade predicts $w = -1$ exactly.*

Proof. $\Lambda = I$ is a fixed geometric constant, determined entirely by π through the cascade's threshold structure ([1], Theorem 9.2). A fixed cosmological constant has $\rho_\Lambda = \text{const}$, giving $p_\Lambda = -\rho_\Lambda$ and $w = p/\rho = -1$ by definition. No time-evolution mechanism exists in the cascade for Λ: the thresholds $d_1 = 19$ and $d_2 = 217$ are permanent features of the Gamma function, not snapshots of a dynamical field. The lapse function $N(4) = 3\pi/8$ is a fixed Gamma-function value, not a cosmological-time-evolving quantity. □

6.1 Gauss–Bonnet stability

The cascade geometry at $d = 5$ has S^3 cross-sections that are totally umbilic: $K_{ij} = -(x/\sqrt{1-x^2})\, h_{ij}$ exactly, so $K_{ij} = (K/3)h_{ij}$. For totally umbilic hypersurfaces, the leading Gauss–Bonnet boundary terms cancel identically:

$$2K_{ij}K^{ij}K - \tfrac{2}{3}K^3 = \tfrac{2}{3}K^3 - \tfrac{2}{3}K^3 = 0.$$

The residual GB term $2K\tilde{R} - 2K_{ij}\tilde{R}^{ij} \propto -24x/(1-x^2)^2$ is odd in x; the cascade measure $(1-x^2)^{3/2}$ is even. The integral vanishes exactly:

$$\int_{-1}^{1} (1-x^2)^{3/2} \cdot \frac{-24x}{(1-x^2)^2}\, dx = -24 \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\, dx = 0.$$

The bulk GB contribution is suppressed by $H_0^2/M_{\rm Pl}^2 \sim 10^{-120}$.

Corollary 6.2. *$w = -1$ receives zero Gauss–Bonnet correction at leading order, by two independent mechanisms: (i) totally umbilic cancellation from the cascade's spherical symmetry; (ii) parity of the residual integrand under symmetric slicing. Both follow from the orthogonality axiom. The prediction is structurally protected, not merely numerically small.*

Remark (Quantitative cosmology)**.** *The cascade's geometric parameters determine the background cosmological observables. The density fractions $\Omega_\Lambda = (\pi-1)/\pi$, $\Omega_m = 1/\pi$ (leading order) or $\Omega_m^{\rm Bott} = 0.31150$ (subleading), and $\Omega_b = 1/(2\pi^2)$ follow from the cascade's sphere-area structure. The Hubble constant $H_0 = 66.78$ km/s/Mpc follows from the Friedmann equation with Part I's observer-corrected $\rho_\Lambda/M^4_{\rm Pl,red} = (2/\pi)\, I$ (where $2/\pi$ is the cube–sphere bridge $V_3^{\rm cube}/\Omega_2$ at the observer's spatial dimension $d = 3$), sitting* 0.9% *below Planck's* 67.4 *at leading order and closing to essentially the Planck central value after the Part 0 Supplement Gram first-order correction; it is incompatible with the SH0ES local measurement of* 73.0*. These parameters predict a sound horizon $r_d \approx 147.75$ Mpc, essentially equal to Planck's* 147.60 *Mpc. The cascade predicts $w = -1$ exactly as a structural theorem; the apparent DESI preference for $w \neq -1$ challenges both the cascade and ΛCDM equally, and cannot be attributed to a ruler mismatch between them.*

Theorem 6.3 (Orthogonality of coupling and vacuum sectors)**.** *The cascade lapse factorises as $N(d) = \sqrt{\pi} \cdot R(d)$. Gauge couplings satisfy $\alpha(d) = N(d)^2/(4\pi) = R(d)^2/4$: the factor of π from squaring $\sqrt{\pi}$ cancels the denominator exactly, leaving $\alpha = R(d)^2/4$: the multiplicative $\sqrt{\pi}$ factor of the lapse drops out exactly. The cosmological constant $\Lambda = I$ is generated by the threshold conditions $p(d) = c_1 = \frac{1}{2}\ln\pi$ and $p(d) = c_2 = \sqrt{\pi}$, which* *are determined by the constant part of $p(d)$—the same $\sqrt{\pi}$ factor—with no dependence on $R(d)$ at the threshold dimensions. The two sectors therefore draw on orthogonal factors of $N(d)$: the multiplicative $\sqrt{\pi}$ of the lapse cancels from α and generates Λ; $R(d)$ determines α and does not enter the threshold conditions. This is why Λ does not run while couplings run.*

Proof. $\alpha(d) = N(d)^2/(4\pi) = \pi R(d)^2/(4\pi) = R(d)^2/4$. The $\sqrt{\pi}$ factor cancels entirely; α depends only on $R(d)$. The thresholds $d_1 = 19$ and $d_2 = 217$ are determined by $p(d) = c_1 = \frac{1}{2}\ln\pi$ and $p(d) = c_2 = \sqrt{\pi}$, where c_1, c_2 are generated from $\sqrt{\pi}$ alone ([1], Theorem 6.5). $\Lambda = I$ is a function of Ω_{d_1} and Ω_{d_2}, which are values of the sphere-area sequence at the threshold dimensions; these values are determined by $\sqrt{\pi}$ through the threshold conditions. At those dimensions, $R(d_1)$ and $R(d_2)$ enter only the d-dependent decay rate, not the threshold conditions themselves. Therefore Λ contains no dependence on $R(d)$ through the threshold structure. □

7 Matter from Extra Dimensions

7.1 The Kaluza–Klein perspective

The cascade produces a 217-dimensional geometry. The 4D observer has integrated out 213 directions via the slicing recurrence. The purpose of this section is to identify the effective stress-energy tensor $T^{\mu\nu}_{\text{eff}}$ that enters the Einstein equation projection identity (Section 8): Lovelock's theorem forces $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G\, T^{\mu\nu}$, and $T^{\mu\nu}_{\text{eff}}$ is the cascade's geometric account of what fills the right-hand side.

Remark (Induced matter via Gauss–Codazzi)**.** *Let $G^{\mu\nu}_{(217)}$ be the Einstein tensor of the cascade's 217-dimensional geometry and $G^{\mu\nu}_{(4)}$ be the 4D projected Einstein tensor. The difference $8\pi G\, T^{\mu\nu}_{\text{eff}} = G^{\mu\nu}_{(4)} - [G^{\mu\nu}_{(217)}]_{4D}$ defines the effective stress-energy seen by the 4D observer, following from the Gauss–Codazzi equations for the embedding $M^4 \subset M^{217}$. This is a structural identification, not a dynamical calculation. Section 14 establishes exact results for the Lorentzian cascade metric: $K_{\mu\nu} = 0$ at every equatorial embedding in the chain $S^3 \hookrightarrow S^4 \hookrightarrow \cdots \hookrightarrow S^{216}$, and the single-step cascade geometry contributes $w = 1/3$ (radiation) in cascade time. The cascade's matter fraction Ω_m is determined by the Bott partition of cascade layers, not by the cascade metric geometry; this is a topological rather than geometric quantity.*

8 Self-Consistency: Einstein Equation as Projection Identity

The 4D observer has: a metric $g_{\mu\nu}$ (Section 4); $\Lambda = I$ (Section 5); $T^{\mu\nu}_{\text{eff}}$ from the extra dimensions (Section 7); and Lovelock's theorem (Section 3). Lovelock then requires $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi G\, T^{\mu\nu}$. This is a projection identity, not a postulate.

Theorem 8.1 (Einstein equation as projection identity)**.** *The Gauss–Codazzi equations for $M^4 \subset M^{217}$, combined with Lovelock's theorem at $d = 4$, yield the Einstein equation with $\Lambda = I$ and $T^{\mu\nu} = T^{\mu\nu}_{\text{eff}}$ as an identity. Every term is geometric output of the cascade; no gravitational postulate is added.*

9 Deriving $d = 4$

Definition 9.1 (Cascade-compatible dimension)**.** *A spacetime dimension d is cascade-compatible if it satisfies:*

(C1) The minimal spinor representation of Spin$(1, d-1)$ is irreducibly complex (not realisable over $\mathbb{R}$).

(C2) The Einstein equation with cosmological constant is the unique divergence-free symmetric rank-2 tensor metric equation (Lovelock uniqueness, Theorem 3.1).

9.1 Why (C1) is forced by the cascade

From Theorem 6.1 of [2]: the orthogonality axiom forces the precession angle $\alpha = \pi/2$, producing the complex structure $J : e_1 \mapsto e_2$, $e_2 \mapsto -e_1$ with $J^2 = -\mathrm{Id}$.

Lemma 9.2 (Complex J requires complex spinors)**.** *For the cascade's state space (carrying J with $J^2 = -Id$) to serve as the spinor representation space of Spin$(1, d-1)$, the minimal spinor representation must be irreducibly complex. Majorana-real spinors are incompatible: by Schur's lemma over $\mathbb{R}$, the commutant algebra of an irreducibly real representation is $\mathbb{R}$, admitting no J with $J^2 = -Id$.*

Proof. If the spinor representation is Majorana (real), the representation space is $\mathbb{R}^n$ and the commutant is $\mathbb{R}$ (Schur's lemma). Any J commuting with all group elements satisfies $J = c \cdot \mathrm{Id}$ for some $c \in \mathbb{R}$. Then $J^2 = c^2\,\mathrm{Id}$. Since $c^2 \geq 0$, we cannot have $J^2 = -\mathrm{Id}$. Contradiction. ☐

9.2 The Clifford algebra classification

The type of the minimal spinor of Spin$(1, d-1)$ is determined by the Clifford algebra Cl$(1, d-1)$, following Bott periodicity with period 8.

d	Cl$(1, d-1)$ structure	Min. spinor	Type	(C1)
2	$M_2(\mathbb{R})$	2 real	Majorana	No
3	$M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$	2 real	Majorana	No
4	$M_2(\mathbb{C}) \otimes_{\mathbb{R}} M_2(\mathbb{R})$	2 complex	Weyl	Yes
5	$M_4(\mathbb{C})$	4 complex	Dirac	Yes
6	$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$	4 complex	Weyl	Yes
7	$M_8(\mathbb{R})$	8 real	Majorana	No
8	$M_{16}(\mathbb{R})$	8 real	Majorana	No

Data from Lounesto [6], Lorentzian signature $(1, d-1)$. Condition (C1) holds for $d \in \{4, 5, 6\}$ in the first Bott period, then $\{10, 11, 12\}$, etc.

9.3 The main theorem

Theorem 9.3 (Unique cascade-compatible dimension)**.** *$d = 4$ is the unique dimension satisfying both (C1) and (C2).*

Proof. From the Clifford classification: (C1) holds for $d \in \{4, 5, 6, 10, 11, 12, \ldots\}$. From Theorem 3.1: (C2) holds if and only if $d = 4$. Intersection: $\{4, 5, 6, 10, \ldots\} \cap \{4\} = \{4\}$. ☐

Corollary 9.4 (Third characterisation of $d = 4$)**.** *The Lorentzian cascade scale factor* $a(t) = \sqrt{1-t^2}$ *gives* $\dot{a} = -t/a$ *and* $\ddot{a} = -1/a^3$*. The Ricci scalar of the* n*-dimensional FRW metric with* $k = 1$ *and this scale factor is:*

$$R^{(n)} = \frac{(n-1)(n-4)}{a^4}.$$

This vanishes if and only if $n = 4$*. The Lorentzian cascade metric is therefore Ricci-flat uniquely in four spacetime dimensions, providing a third independent characterisation of* $d = 4$ *complementing Lovelock uniqueness (C2) and complex spinor compatibility (C1).*

Proof. For n-dimensional FRW with $k = 1$: $R_{tt} = -(n-1)\ddot{a}/a = (n-1)/a^4$ and $R_{ij} = [(n-2)(k+\dot{a}^2)/a^2 + \ddot{a}/a]\, g_{ij} = (n-3)/a^4 \cdot g_{ij}$. The Ricci scalar: $R^{(n)} = g^{tt}R_{tt} + g^{ij}R_{ij} = -(n-1)/a^4 + (n-1)(n-3)/a^4 = (n-1)(n-4)/a^4$. At $n = 4$: $R^{(4)} = 3 \cdot 0/a^4 = 0$. For $n \neq 4$: $(n-1)(n-4) \neq 0$. $\square$

Remark. *Without (C1):* $d = 4$ *is selected by Lovelock alone, but the quantum state space may be incompatible with* $J^2 = -Id$*. Without (C2): dimensions 5, 6, 10,... have complex spinors but non-unique gravity with free coupling constants. Only* $d = 4$ *is simultaneously quantum-compatible, gravitationally unique, and Ricci-flat under the cascade scale factor.*

Remark (Division algebra interpretation)**.** *The cascade forces the quantum amplitude algebra to be* $\mathbb{C}$ *(real dimension 2). The associative normed division algebras over* $\mathbb{R}$ *are* $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ *(Hurwitz theorem [7];* $\mathbb{O}$ *is non-associative and excluded from quantum mechanics by associativity of sequential measurements). The minimal associative algebra over* $\mathbb{C}$ *is* $\mathbb{H}$ *(quaternions), of real dimension 4. The spacetime dimension equals the real dimension of the next associative division algebra:* $d = \dim_{\mathbb{R}} \mathbb{H} = 4$.

10 Lorentzian Signature from the Propagator

The cascade geometry is Euclidean: $ds_E^2 = dx^2 + (1-x^2)\, d\sigma_{d-1}^2$ with positive-definite metric. The cascade propagator is $K(x) = |K|\, e^{-i\lambda x}$ with $\lambda > 0$—already oscillatory for real x. These are two descriptions of the same object under the physical identification hypothesis. The Lorentzian signature is the unique metric sign consistent with both descriptions simultaneously.

10.1 The propagator characterises the signature

Lemma 10.1 (Spectral gap is positive)**.** *The cascade spectral gap satisfies* $\lambda_\infty = \frac{1}{4}\psi^{(1)}(\sigma/2) > 0$ *for all* $\sigma > 0$*, where* $\psi^{(1)}$ *is the trigamma function. This follows from* $p'(d) = \frac{1}{4}\psi^{(1)}((d+1)/2) > 0$ *(strict monotonicity of* p*).*

From Theorems 6.1 and 7.1 of [2]: the forced precession $\alpha = \pi/2$ gives the cascade propagator:

$$K(t) = |K| \cdot e^{-iHt}, \qquad H = \lambda_\infty > 0,$$

where $t = x$ is the physical time identified with the slicing coordinate.

Theorem 10.2 (Lorentzian signature from orthogonality)**.** *The spacetime metric of the 4D cascade observer has signature* $(-,+,+,+)$.

Proof. **Temporal component.** Lorentzian signature means $g_{tt} < 0$. This is equivalent, by definition, to time evolution being generated by a self-adjoint Hamiltonian with real eigenvalues, giving unitary oscillatory evolution e^{-iHt}. The cascade propagator (from Theorems 6.1 and 7.1 of [2]) is $K(t) = |K| \cdot e^{-iHt}$ with $H = \lambda_\infty > 0$, which is oscillatory with $|K| = \text{const}$. Since the cascade propagator is oscillatory, the identification hypothesis (Definition 2.1) gives $g_{tt} < 0$.

Spatial components. The cascade's state space is a complex Hilbert space $\mathcal{H}$ ([2]). The Hilbert space inner product is positive-definite: $\langle\psi|\psi\rangle > 0$ for all $|\psi\rangle \neq 0$. The inner product on equal-time surfaces requires a positive-definite spatial volume element $\sqrt{\det g^{(3)}} > 0$, which holds if and only if $g_{ij} > 0$. The signature is $(-, +, +, +)$. □

Remark (Euclidean geometry, Lorentzian physics)**.** *The cascade's underlying geometry is Euclidean; the physical spacetime is Lorentzian. These are not contradictory. The connection is through the propagator's character: the cascade's Euclidean geometry produces an oscillatory propagator (because of the forced precession), and oscillatory propagator character is the physical content of Lorentzian signature. The Euclidean geometry generates Lorentzian physics through the dynamics, not through analytic continuation of the metric.*

Remark (Three consequences of $\psi^{(1)} > 0$)**.** *The strict positivity $\psi^{(1)}(x) > 0$ simultaneously establishes: (a) unique natural zero of $p(d)$ in [1]: requires $p'(d) > 0$; (b) positive spectral gap $\lambda_\infty > 0$; (c) Lorentzian signature: $\lambda_\infty > 0$ makes the propagator oscillatory, requiring $g_{tt} < 0$ (Theorem 10.2).*

11 Scale Factors: G and $\hbar$

The cascade produces dimensionless quantities. Physical gravity requires G (Newton) and $\hbar$ (Planck). Both enter as unit-matching constants when geometric rates are identified with physical observables:

- G: proportionality between 4D curvature and the effective stress-energy from the extra dimensions. Set by the embedding geometry of $M^4 \subset M^{217}$.
- $\hbar = (\pi/2)/\Delta t$: the ratio of the forced precession angle to the physical time increment (Section 9 of [2]). The angle $\pi/2$ is derived; Δt is not.

12 No Gravitons: Metric Degrees of Freedom without Second Quantisation

The cascade does not produce gravitons. The 4D metric $g_{\mu\nu}$ is a property of the cascade state $|\Psi\rangle \in S^{d-1}$ (Paper II = III, Theorem 6.1): different states produce different metrics, superpositions of states produce superpositions of metrics, and the Born rule assigns probabilities to metric outcomes. The metric is never promoted to an operator. There is no spin-2 particle mediating gravitational interactions.

In the standard framework, a massless spin-2 field in d spacetime dimensions has $d(d-3)/2$ independent components. At $d = 4$ this gives 2—the plus and cross polarisations of linearised GR. In the cascade, these are not particle states but the two independent directions in which the 4D metric can be perturbed while satisfying the linearised Einstein equation. The distinction matters: perturbation modes describe the geometry's response to small disturbances; they do not require quantisation into particles.

The higher-dimensional metric has 217 × 214/2 = 23,219 independent components. From the 4D observer's perspective, these appear as a Kaluza–Klein tower of massive fields on adjacent shells, with mass scale set by the compactification radii $R_{\text{eff}}(d) = 1/\sqrt{d+3}$ (Paper I, Theorem 4.2). These are geometric degrees of freedom of the cascade—effective fields in the 4D projection—not fundamental particles. The cascade predicts no graviton will be detected, for the same structural reason it predicts no supersymmetric partners, no extra gauge bosons, and no dark matter particles: the topology has no mechanism to produce them.

13 The Complete Derivation

The full chain from the orthogonality axiom to the metric of general relativity:

Step	Result	Source
Orthogonality $\Rightarrow \sqrt{\pi}$	Cascade constant	[1], Thm 3.1
$\sqrt{\pi} \Rightarrow I \approx 10^{-120}$	Cascade invariant	[1], Thm 9.2
Orthogonality $\Rightarrow J^2 = -\text{Id}$	Complex structure	[2], Thm 6.4
$J^2 = -\text{Id} \Rightarrow$ complex spinors (C1)	Schur + Clifford	Section 9
(C1)$\cap$Lovelock (C2)$\Rightarrow d = 4$	Dimension theorem	Thm 9.3
$R^{(n)} = (n-1)(n-4)/a^4 \Rightarrow R^{(4)} = 0$	Third char. of $d = 4$	Cor 9.4
$d = 4 \Rightarrow$ FRW metric + $\Lambda = I$	Cascade geometry	§4, §5
Physical identification hypothesis	Cascade geom = physics	Def 2.1
Hypothesis + $\lambda_\infty > 0 \Rightarrow g_{tt} < 0$	Propagator	Thm 10.2
$d = 4 + (-,+,+,+) \Rightarrow$ Einstein eq.	Lovelock	Section 8
Fixed $\Lambda = I \Rightarrow w = -1$	Definition	Section 6
GB corrections vanish	Two mechanisms	Section 6
$N = \sqrt{\pi}\,R \Rightarrow$ orthogonal sectors	Couplings vs vacuum	Thm 6.4
Extra dims $\Rightarrow T^{\mu\nu}$	Gauss–Codazzi	Section 7

13.1 Full result table

Physical result	Cascade provides	Classical theorem
Complex QM, Born rule, $\hbar$	Complex structure, propagator	Sphere geometry, [2]
Cosmological constant $\Lambda = I$	Cascade invariant I	Fixed, not free
$d = 4$ spacetime dimensions	Complex spinors + gravity	Clifford, Lovelock
$R^{(4)} = 0$: Ricci-flat cascade metric	Lorentzian scale factor	GR, Corollary 9.4
Lorentzian signature $(-,+,+,+)$	Propagator e^{-iHt}, $H > 0$	Path integral identity
Einstein equation	Metric + $\Lambda = I$	Lovelock
FRW cosmology	Cascade foliation	Geometry
$w = -1$ dark energy	Fixed $\Lambda = I$; GB vanishes	Section 6
$\Lambda \approx 10^{-120} M_{\text{Pl}}^4$	Sphere-area hierarchy	[1], Thm 10.1
Orthogonal coupling and vacuum sectors	$N = \sqrt{\pi}\,R$ factorisation	Thm 6.4
$K_{\mu\nu} = 0$: totally geodesic embedding	Lorentzian cascade metric	Section 14

14 Supplement: Asymptotic Compactification and the Cascade Metric

This section integrates the compactification results of [1], Section 4 into the gravitational framework, and presents the exact Lorentzian Gauss–Codazzi analysis.

14.1 The compactification radius of each dimension

Theorem 14.1 (Compactification radius; [1], Theorem 4.2)**.** *At each slicing step d, the integrated-out direction retains an effective radius*

$$R_{\mathit{eff}}(d) = \frac{1}{\sqrt{d+3}},$$

derived exactly from $\langle x^2 \rangle = B(3/2, d/2+1)/B(1/2, d/2+1) = 1/(d+3)$.

The compactification is asymptotic: $R_{\text{eff}} \to 0$ as $d \to \infty$, but $R_{\text{eff}} \neq 0$ at any finite d. There is no sharp boundary, no horizon, no topology change—only exponential suppression.

d	R_{eff}	$N(d)$	Physical role
4	0.37796	1.17810	Observer dimension
5	0.35355	1.06667	S^3 boundary layer
7	0.31623	0.91429	Volume maximum d_0
12	0.25820	0.70870	SU(3) layer
13	0.25000	0.68198	SU(2) breaking
19	0.21320	0.56755	First threshold d_1
217	0.06742	0.16997	Second threshold d_2

Remark (KK mass scale)**.** *The KK mass scale at dimension d is $M_{KK}(d) \sim 1/R_{\mathit{eff}}(d) = \sqrt{d+3}$ in cascade units. No free parameter adjusts the mass scale.*

14.2 Boundary dominance and the primacy of sphere areas

Theorem 14.2 (Boundary dominance; [1], Theorem 4.4)**.** *$\Omega_{d-1}/V_d = d$ for all $d \geq 1$.*

At $d = 4$: $\Omega_3/V_4 = 4$, so the boundary S^3 carries $4/5 = 80\%$ of the content of B^4. At $d = 217$: essentially all content is on the boundary. The cascade's content is its boundaries, increasingly so at each step.

14.3 The observer on S^3

The 4D observer exists on the boundary S^3 of the $d = 5$ layer. Theorem 14.3 shows this boundary carries $5/6 \approx 83\%$ of the $d = 5$ content. The observer's physics is a boundary theory, not because holography is assumed, but because the boundary dominates the volume at every cascade layer.

The S^3 horizon of a $d = 5$ Schwarzschild black hole has topology S^3, matching the 3 spatial dimensions. The cascade's $d = 5$ layer provides this horizon. The de Sitter horizon area $A = 12\pi/\Lambda \sim 10^{120}$ in Planck units is the cascade hierarchy inverted: $\Omega_7/\Omega_{217} \approx 10^{121}$.

Remark (The cosmological constant as inverse boundary area)**.** *With boundary dominance, the identification $\Lambda = I$ acquires a geometric interpretation: Λ measures the inverse boundary area of the cascade's distinguished layers. The smallness of Λ reflects the largeness of the boundary—a high-capacity boundary (S^3 with $\Omega_3 = 2\pi^2 \approx 19.7$) encodes an enormous cascade (213 compactified directions), giving $\Lambda = I \approx 10^{-120}$.*

14.4 The Lorentzian Gauss–Codazzi analysis

We install the Lorentzian cascade metric directly from Theorem 10.2, without analytic continuation. The n-dimensional Lorentzian cascade metric is:

$$g^L = -dt^2 + (1 - t^2)\, d\sigma^2_{n-1}, \qquad a(t) = \sqrt{1 - t^2}.$$

With $\dot{a} = -t/a$ and $\ddot{a} = -1/a^3$, two exact identities hold:

$$\frac{k + \dot{a}^2}{a^2} = \frac{1}{a^4}, \qquad \frac{\ddot{a}}{a} = -\frac{1}{a^4}.$$

Theorem 14.3 (Lorentzian cascade Ricci tensor)**.** *For the n-dimensional Lorentzian cascade metric:*

$$R^{(n)}_{tt} = \frac{n-1}{a^4}, \qquad R^{(n)}_{ij} = \frac{n-3}{a^4}\, g_{ij}, \qquad R^{(n)} = \frac{(n-1)(n-4)}{a^4},$$

$$G^{(n)}_{tt} = \frac{(n-1)(n-2)}{2a^4}, \qquad G^{(n)}_{NN} = -\frac{(n-2)(n-5)}{2a^4}.$$

Proof. Direct computation using the FRW Ricci tensor formulae with $k = 1$, $n-1$ spatial dimensions, and the cascade identities $\dot{a}^2 + k = 1/a^2$, $\ddot{a}/a = -1/a^4$: $R_{tt} = -(n-1)\ddot{a}/a = (n-1)/a^4$; $R_{ij} = [(n-2)(\dot{a}^2+k)/a^2 + \ddot{a}/a]\, g_{ij} = [(n-2)/a^4 - 1/a^4]\, g_{ij} = (n-3)/a^4 \cdot g_{ij}$. The Ricci scalar: $R^{(n)} = g^{tt}R_{tt} + g^{ij}R_{ij} = -(n-1)/a^4 + (n-1)(n-3)/a^4 = (n-1)(n-4)/a^4$. For the normal-normal Einstein component at the equatorial embedding $\psi = \pi/2$ in $S^{n-1} \subset S^n$: $N_\psi = (1-t^2)^{-1/2}$, giving $R^{(n)}_{NN} = N_\psi N_\psi R^{(n)}_{\psi\psi} = (n-3)/a^4$, and $G^{(n)}_{NN} = R^{(n)}_{NN} - \frac{1}{2} g_{NN} R^{(n)} = (n-3)/a^4 - (n-1)(n-4)/(2a^4) = -(n-2)(n-5)/(2a^4)$. □

Theorem 14.4 (Totally geodesic embedding)**.** *The 4D Lorentzian cascade spacetime $M^4 = \mathbb{R} \times S^3$ embeds in $M^5 = \mathbb{R} \times S^4$ as a totally geodesic submanifold: $K_{\mu\nu} = 0$. The same holds at every equatorial embedding in the chain $S^3 \hookrightarrow S^4 \hookrightarrow \cdots \hookrightarrow S^{216}$.*

Proof. Parametrise $S^4 = \{(\psi, \theta_i)\}$ with $d\sigma_4^2 = d\psi^2 + \sin^2\psi\, d\sigma_3^2$. The 4D submanifold sits at $\psi = \pi/2$. The spacelike unit normal is $N = (1-t^2)^{-1/2}\partial_\psi$. For the temporal component: $g_{tt} = -1$ is independent of ψ, so $K_{tt} = 0$. For the spatial components: $\partial_\psi g_{\theta_i\theta_j} = 2(1-t^2)\sin\psi\cos\psi\, \bar{g}_{ij}$; at $\psi = \pi/2$: $\cos(\pi/2) = 0$, so $K_{ij} = 0$. Therefore $K_{\mu\nu} = 0$ identically. The argument applies at every equatorial embedding by identical reasoning. □

Corollary 14.5 (Hamiltonian constraint)**.** *The Hamiltonian constraint ${}^{(4)}R = -2G^{(5)}_{NN}$ is satisfied identically: $0 = 0$ at every cascade level. The cascade embedding chain is geometrically self-consistent.*

Proof. The scalar Gauss equation for a hypersurface with $K_{\mu\nu} = 0$ gives ${}^{(4)}R = {}^{(5)}R - 2R^{(5)}_{NN}$. Since $R^{(5)}_{NN} = G^{(5)}_{NN} + \frac{1}{2}R^{(5)}$, substituting: ${}^{(4)}R = R^{(5)} - 2(G^{(5)}_{NN} + \frac{1}{2}R^{(5)}) = -2G^{(5)}_{NN}$. From Theorem 14.5 at $n = 5$: $G^{(5)}_{NN} = -(5-2)(5-5)/(2a^4) = 0$. From Corollary 9.4 at $n = 4$: $R^{(4)} = 0$. Therefore $0 = -2 \cdot 0 = 0$. ✓ □

Theorem 14.6 (Effective equation of state of the cascade metric)**.** *The 4D Lorentzian cascade metric has effective equation of state* $w = 1/3$ *(radiation) in cascade time, at every cascade level.*

Proof. From Theorem 14.5 at $n = 4$ (using $R^{(4)} = 0$, so $G^{(4)}_{\mu\nu} = R^{(4)}_{\mu\nu}$): $G^{(4)}_{tt} = R^{(4)}_{tt} = 3/a^4$; $G^{(4)}_{ij} = R^{(4)}_{ij} = (4-3)/a^4 \cdot g_{ij} = g_{ij}/a^4$. Identifying $\rho_{\text{eff}} \propto G^{(4)}_{tt} = 3/a^4$ and $p_{\text{eff}} \propto G^{(4)}_{ij}/g_{ij} = 1/a^4$, the effective equation of state is:

$$w_{\text{eff}} = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = \frac{1/a^4}{3/a^4} = \frac{1}{3}.$$

□

Remark (Consequences for the matter fraction)**.** *Theorems 14.6 and 14.8 establish that the geometric Gauss–Codazzi route to the cascade matter fraction* Ω_m *is definitively closed. Three independent exact results confirm this:*

1. $K_{\mu\nu} = 0$ *(Theorem 14.6) eliminates the standard extrinsic curvature mechanism for KK matter generation. The cascade slicing geometry does not source matter via this channel.*
2. $w = 1/3$ *(Theorem 14.8) shows the cascade metric contributes radiation, not matter (*$w = 0$*) or dark energy (*$w = -1$*), in cascade time. No splitting of the cascade integrand into core and tail produces the combination (*$w = -1$, $w = 0$*) consistent with the Friedmann equation.*
3. *The Hamiltonian constraint telescopes identically (Corollary 14.7):* ${}^{(4)}R = -2G^{(5)}_{NN} = 0 = 0$ *at every level, providing geometric consistency without yielding matter content.*

The cascade matter fraction Ω_m *is determined by the Bott partition of cascade layers—the sphere-area fraction of non-trivial-phase layers (*$d \bmod 8 \in \{5, 6\}$*). This requires three inputs: the propagator phase classification ([2], Corollary 6.5; period-8 refinement in this paper, Section 9), which establishes the topological partition; the fermion generation structure of the Bott partition ([3]), which identifies non-trivial-phase layers with matter via the hairy ball zeros and observed fermion generations; and the uniqueness of sphere areas as cascade content (Corollary 3.2 of [1]), which forces the energy assignment to be proportional to* Ω_{d-1} *with a universal constant. The matter fraction is a topological rather than geometric quantity; it does not require the cascade metric and is unaffected by the Gauss–Codazzi results above. The explicit derivation is given in the cosmological companion.*

14.5 The information paradox

The standard black hole information paradox asks how information is preserved when it crosses a horizon. In the cascade:

1. **No sharp horizon.** The compactification at each step is asymptotic: $R_{\text{eff}} = 1/\sqrt{d+3} > 0$ for all finite d. The weight function $(1-x^2)^{d/2}$ has support everywhere on $(-1, 1)$. No direction is fully removed; no information is lost.

2. **Unitary dynamics.** The discrete propagator ([2], Theorem 7.3) preserves the phase structure. Each slicing step is a unitary operation on the Hilbert space.
3. **Apparent irreversibility is asymptotic.** From above ($d+1$ dimensions), a slicing step looks irreversible: the direction's contribution is suppressed to $O(1/\sqrt{d})$. From below (d dimensions), the boundary carries the full content (boundary dominance). Both descriptions are correct; they describe the same asymptotic process from different sides.

There is no paradox because there was never a sharp boundary to lose information across—only asymptotic compactification. No firewall, no information loss, no complementarity principle needed.

15 What This Paper Does Not Do

- It does not derive Newton's constant G or the local solutions of GR.
- It does not derive the Standard Model gauge group SU(3) × SU(2) × U(1).
- It does not unify QM and GR in the sense of a quantum gravity theory; Paper II = III shows this is unnecessary because both are projections of the same cascade state.
- It requires one physical identification hypothesis beyond pure mathematics (Definition 2.1): that the cascade geometry is indistinguishable from our universe. Without it the series is pure mathematics.
- It uses no result from physics: all remaining inputs are the orthogonality axiom and classical mathematical theorems (Lovelock, Clifford–Bott, Hurwitz).

16 Open Questions

1. **The Standard Model.** The tower $\{7, 12, 198\}$ (dimensions below d_0, between d_0 and d_1, between d_1 and d_2) provides natural sub-bundles. The number 12 matches the generator count of SU(3) × SU(2) × U(1).
2. **Quantum gravity.** Resolved in Paper II = III: the quantum and gravitational projections share the same source, propagator, and state space. Double uniqueness (Gleason + Lovelock at $d = 4$) proves mutual consistency. The metric is a state property; gravity was always quantum in the cascade.
3. **Black holes.** Resolved in Paper II = III: boundary dominance gives $S = A/4$ at $d = 4$, and the Hawking temperature $T = 1/(8\pi M)$ follows from Birkhoff's theorem with no semiclassical input. Remaining open questions (thermal spectrum, Page curve, Kerr/RN entropy) are listed there.
4. **Newton's constant.** The cascade produces dimensionless quantities; G enters as a unit-matching constant from the embedding $M^4 \subset M^{217}$. Deriving G from the cascade's KK mass spectrum (with compactification radii $R_{\text{eff}}(d) = 1/\sqrt{d+3}$ now determined) is the next natural step.

5. **The cascade Friedmann equation.** The cascade metric (Section 4) foliates a 217-dimensional geometry onto a 4D FRW slice. The projected Friedmann equation should follow from the Gauss–Codazzi equations applied to the $M^4 \subset M^{217}$ embedding. The Lorentzian Gauss–Codazzi analysis (Section 14.4) establishes that the cascade metric contributes radiation ($w = 1/3$) and that $K_{\mu\nu} = 0$ eliminates the extrinsic curvature mechanism. The matter and vacuum energy content of the 4D Friedmann equation enters via the Bott partition of cascade layers (Remark 14.9), not via the cascade metric geometry. A complete derivation of the Friedmann equation connecting the cascade's KK structure to the physical expansion history remains open.

6. **The Hubble tension.** The cascade's parameter triple $(H_0, \Omega_m, \Omega_b)$ gives a sound horizon smaller than the Planck-inferred value. The cascade's H_0 sits between the two principal measurements. The full cosmological derivation is a separate programme.

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