The Cascade Series — Part II

Quantum Mechanics from the Cascade: Effective Theory of a 4-Dimensional Observer in the Sphere-Area Geometry

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Abstract

The cascade series tests one hypothesis: the infinite-dimensional unit ball, descended to four dimensions, is indistinguishable from our universe. The companion paper [1] derived a geometric invariant $I = 1.0990 \times 10^{-120}$ from the sphere-area cascade using only orthogonality as input. Here we show that a 4-dimensional observer embedded in the cascade geometry recovers the full structural framework of quantum mechanics—including the Born rule, complex amplitudes, non-commutativity, the Schrödinger equation, entanglement, and Bell inequality violation—without quantum postulates.

The argument proceeds in six steps, each a theorem about spheres and Beta functions. (1) The cascade's boundary dominance ($\Omega_{d-1}/V_d = d$) fixes the state space as the unit sphere S^{d-1}; projection onto a 4D subspace produces Gaussian statistics via concentration of measure. (2) The cascade's orthogonal slicing structure induces measurement bases; the commutator of rank-1 projections at angle θ is $\|[P, Q]\| = \frac{1}{2}|\sin 2\theta|$, vanishing at $\theta \in \{0, \pi/2, \pi\}$. (3) The Born rule $p = \cos^2\theta$ is derived from the geometry of the spherical cap, the slicing integrand $(1 - z^2)^{(d-2)/2}$, and Parseval's identity $\sum(u \cdot e_j)^2 = 1$; no probability axiom enters. (4) The precession angle $\alpha = \pi/2$ between consecutive slicing axes is forced by the same orthogonality axiom that forces $\sqrt{\pi}$ in the slicing recurrence; two quarter-turns give $J^2 = -\mathrm{Id}$, producing complex amplitudes. The phase–obstruction lockstep (Corollary 6.6) follows: the propagator phase is imaginary if and only if S^{d-1} carries a hairy ball zero, both controlled by the parity of d. The cascade's classification has period 4; the Clifford refinement to period 8 is established in Part III. (5) Time is the cascade's slicing direction. The discrete propagator $K = \prod |L(j)| \cdot i^{D-d'}$ is exact; the Schrödinger equation $i\dot{\psi} = \mathcal{H}\psi$ with $\mathcal{H} = (1 - N)/N^2$ (a Gamma function ratio) is the effective description for a 4D observer who cannot resolve individual cascade steps. Each step compactifies one direction to $R_{\text{eff}} = 1/\sqrt{d+3}$ (from the Beta function), producing geometric decoherence: the decoherence factor $D = \exp(-\Delta x^2/4R_{\text{eff}}^2)$ identifies the compactification radius as the coherence length. Over the full 213-step descent, the cascade retains 93.9% coherence; the 6.1% eigenvalue deficit is the inter-layer coupling that drives the descent corrections in the Part 0 Supplement. (6) Iterated slicing gives a tensor product structure with generic entanglement, and the CHSH correlator computed from the Born rule on S^7 gives $S = 2\sqrt{2}$, violating the classical bound of 2.

The value of $\hbar$ is not derived; it enters as the ratio of the forced precession scale $\pi/2$ to the physical time increment, encoding only the unit matching between geometric and physical time. No physical law or free geometric parameter beyond the observer's dimensionality enters the derivation.

Contents

1 Introduction and Summary of the Foundation

In [1], the sphere-area cascade was derived from a single axiom: orthogonality. Starting from the infinite-dimensional unit ball—which has zero volume, zero surface area, and no interior—the cascade compresses orthogonal directions one at a time, descending from $d = \infty$. The unique dimension-independent constant in the slicing recurrence is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, forced by the quarter-turn integral $B(\frac{1}{2}, \cdot)$. This constant enters in two roles (multiplicative factor and bare decay rate), producing two distinguished dimensions $d_1 = 19$ and $d_2 = 217$, and a cascade invariant $I = 9\,\Omega_{19}\,\Omega_{217}/\pi^2 = 1.0990 \times 10^{-120}$.

Corollary 3.2 of [1] establishes that sphere areas are the unique independent cascade quantities: every other cascade object is derived from sphere areas, and any identification mapping cascade content to energy must do so with a universal constant. Section 8 of this paper develops the full compactification interpretation: each slicing step sends one direction's effective radius to $R_{\text{eff}}(d) = 1/\sqrt{d+3}$, and the boundary-to-volume ratio satisfies $\Omega_{d-1}/V_d = d$ for all $d \geq 1$. The cascade's content is its boundaries; sphere areas, not volumes, are the primary objects.

The present paper asks: what does the cascade geometry look like from inside? Specifically, if an observer is restricted to a 4-dimensional cross-section of the full cascade structure, what measurement theory does that observer use? We show the answer is quantum mechanics.

The logical structure is:

- *Foundation [1]*: Orthogonality → $\sqrt{\pi}$ → slicing recurrence → cascade → tower $(d_0, d_1, d_2) \to I \approx 10^{-120}$. Each step is an asymptotic compactification with $R_{\text{eff}} = 1/\sqrt{d+3}$.
- *This paper*: Cascade geometry + 4D observer → Hilbert space + Gaussian measure + Born rule ($p = \cos^2\theta$) + complex amplitudes ($J^2 = -\text{Id}$) + discrete propagator + decoherence ($D = e^{-\Delta x^2/4R_{\text{eff}}^2}$) + Bell violation ($S = 2\sqrt{2}$) = QM.

The derivation does not explain why we observe four dimensions. That is an empirical input, just as in [1]. What we show is that, given a 4D perspective, the cascade's internal structure reproduces the quantum formalism.

2 The Observer's Arena

2.1 State space is the unit sphere

Quantum states are unit vectors in a Hilbert space $\mathcal{H}$. The set of pure states is the unit sphere $S(\mathcal{H})$. In finite dimension d, this is $S^{d-1} \subset \mathbb{R}^d$ (or $S^{2d-1} \subset \mathbb{C}^d$ after complexification). The cascade geometry provides exactly this arena: the unit sphere S^{d-1} at each level d of the descent. A 4D observer—one who can access only 4 orthogonal directions—sees the projection of S^{d-1} onto a 4-dimensional subspace. This projected space is the observer's effective state space.

2.2 Why the cascade fixes the arena

Standard quantum mechanics postulates a Hilbert space but does not derive it. The cascade provides the derivation: the arena is the unit sphere because the cascade is a sequence of volume-preserving orthogonal compressions of the unit ball. Every state accessible to any observer at any dimension lies on a unit sphere. The constraint $|\psi|^2 = 1$, which in QM is the normalisation postulate, is here a theorem: the cascade operates on B^d, whose boundary is S^{d-1}.

2.3 Boundary dominance and the primacy of the sphere

The arena is the unit sphere rather than the unit ball for a reason that [1], Section 3 makes precise.

Theorem 2.1 (Boundary dominance; Theorem 3.1 of [1]). *$\Omega_{d-1}/V_d = d$ for all $d \geq 1$.*

At $d = 4$: $\Omega_3/V_4 = 4$, so the boundary S^3 carries a fraction $d/(d+1) = 4/5 = 80\%$ of the content of B^4. At the cascade dimensions $d_1 = 19$ and $d_2 = 217$, the fraction on the boundary is $19/20 = 95\%$ and $217/218 \approx 99.5\%$ respectively. The cascade's content is its boundaries, increasingly so at each step. The observer's state space is the sphere because the sphere carries nearly all the geometric content available to the observer. This is not an axiom; it is a consequence of the Gamma function identity $\Omega_{d-1}/V_d = 2\pi^{d/2}/[\Gamma(d/2)] \div \pi^{d/2}/[\Gamma(d/2+1)] = d$.

Remark 2.2 (Layered boundary dominance). *The cascade stacks boundary-dominant steps: the d_2-sphere's boundary encloses the d_1-sphere's boundary, which encloses the d_0-sphere's boundary, which encloses the observer's S^3. At each step, the boundary S^{d-1} carries a fraction $d/(d+1)$ of the content. The cascade is a nested sequence of shells, each carrying nearly all the geometric content of its level, with the observer on the innermost shell.*

3 Gaussian Statistics from Projection

3.1 Concentration of the slicing integrand

The slicing integrand $f_d(x) = (1 - x^2)^{d/2}$ that defines the cascade recurrence is itself approximately Gaussian. This is the concrete mechanism underlying the projection theorem that follows.

Lemma 3.1 (Gaussian concentration of the integrand). *The slicing integrand $f_d(x) = (1 - x^2)^{d/2}$ satisfies:*

(a) $f_d(x) = \exp(-dx^2/2 + O(dx^4))$, approximating a Gaussian with standard deviation $\sigma(d) = 1/\sqrt{d}$;

(b) the half-maximum width is $\sqrt{2\ln 2}/\sqrt{d}$;

(c) as $d \to \infty$, $f_d \to \delta(x)$.

Proof. Expanding: $\ln(1-x^2) = -x^2 - x^4/2 - x^6/3 - \cdots$ for $|x| < 1$. Therefore $(d/2)\ln(1-x^2) = -dx^2/2 + O(dx^4)$. The half-maximum occurs at $x_{1/2} = \sqrt{1 - 2^{-2/d}} \approx \sqrt{2\ln 2}/\sqrt{d}$. As $d \to \infty$, $x_{1/2} \to 0$ and $\int f_d \, dx \approx \sqrt{2\pi/d} \to 0$, confirming convergence to $\delta(x)$. □

This lemma is the elementary origin of all Gaussian structure in the cascade. The projection theorem (below) elevates it to a distributional statement on the unit sphere; the compactification results (Section 8) use it to define the effective radius of each temporal step.

3.2 The projection theorem

Theorem 3.2 (Gaussian emergence)**.** *Let x be uniformly distributed on $S^{d-1}(\sqrt{d}) \subset \mathbb{R}^d$. Let $\pi_k : \mathbb{R}^d \to \mathbb{R}^k$ be projection onto any k-dimensional subspace, with k fixed and $d \to \infty$. Then $\pi_k(x)$ converges in distribution to $\mathcal{N}(0, I_k)$, the standard k-dimensional Gaussian.*

This is a consequence of the Poincaré limit theorem (see [2, 3]). The rate of convergence is $O(1/d)$, so for the cascade dimensions $d_1 = 19$, $d_2 = 217$, the approximation is already excellent for $k = 4$.

3.3 The cascade's Gaussian is the QM wavefunction

The Gaussian that emerges from projection is structurally identical to the ground-state wavefunction of the quantum harmonic oscillator: $\psi_0(x) = \pi^{-1/4} \exp(-x^2/2)$. The normalisation constant $\pi^{-1/4}$ involves exactly the $\sqrt{\pi}$ from the cascade. In standard QM this constant is obtained by requiring $\int |\psi|^2 dx = 1$ with the Gaussian integral $\int \exp(-x^2) dx = \sqrt{\pi}$. In the cascade, $\sqrt{\pi}$ is not a normalisation convention: it is the unique dimension-independent constant forced by orthogonal compression (Theorem 3.1 of [1]). The cascade derives what QM postulates.

3.4 Why Gaussians are universal

The cascade provides a structural explanation for why Gaussians dominate quantum physics. Any 4D observer embedded in a high-dimensional unit-sphere geometry necessarily sees Gaussian statistics, because projection from high to low dimensions produces Gaussians. This is a geometric statement: the shape of the wavefunction is fixed by the shape of the arena.

4 Orthogonal Measurement Bases

4.1 Slicing induces measurement

In the cascade, each step selects an orthogonal direction and integrates over it. From the perspective of a 4D observer, this operation has a natural interpretation: choosing a direction along which to slice is choosing an observable to measure. The act of slicing—projecting the $(d+1)$-ball onto the equatorial d-ball—discards information about the perpendicular coordinate. This is measurement: extracting a definite value along one axis at the cost of information about the complementary axis.

The compactification results of Section 8 sharpen this interpretation. Each slicing step does not annihilate the integrated-out direction; it suppresses it to an effective radius $R_{\text{eff}}(d) = 1/\sqrt{d+3}$ (Theorem 8.1). The weight function $(1-x^2)^{d/2}$ remains positive on $(-1, 1)$ for all finite d. Measurement in the cascade is asymptotic suppression, not sharp projection: the complementary coordinate is exponentially suppressed but never exactly zero. This is a geometric realisation of the fact that quantum measurement does not destroy the unmeasured degree of freedom—it renders it inaccessible at scale R_{eff}.

Theorem 4.1 (Orthogonal bases from slicing)**.** *The cascade's slicing structure, restricted to a 4D subspace, determines a family of orthogonal bases for $\mathbb{R}^4$. Each choice of slicing axis within the 4D subspace defines a measurement basis. Two measurements are complementary if and only if their slicing axes are orthogonal in $\mathbb{R}^4$.*

Proof. The slicing integral decomposes B^{d+1} along a chosen axis e. Restricted to a 4D subspace, the choice of e selects a direction in $\mathbb{R}^4$. Different choices of e give different decompositions of $\mathbb{R}^4$ into (axis)×(3-ball). Two axes e_1, e_2 are orthogonal if and only if the corresponding decompositions share no common axis, which is the definition of complementary observables. $\square$

4.2 Non-commutativity from sequential slicing

Slicing along axis e_1 then e_2 is not the same as slicing along e_2 then e_1, because the first slice suppresses information about the first axis before the second slice extracts information about the second. The cascade's slicing along axis e extracts the component $z = x \cdot e$ and suppresses the remainder. The extraction is an orthogonal projection: $P_e(x) = (x \cdot e)\, e$, a rank-1 operator with $P_e^2 = P_e$.

Theorem 4.2 (Commutator from slicing angle)**.** *Let e_1, e_2 be two slicing axes in the 4D subspace with $e_1 \cdot e_2 = \cos\theta$. The commutator of the corresponding projections satisfies*

$$\|P_{e_1}P_{e_2} - P_{e_2}P_{e_1}\| = \sin\theta\,|\cos\theta|. \tag{1}$$

This vanishes if and only if $\theta = 0$, $\pi/2$, or π. The order-dependence of sequential slicing is maximal at $\theta = \pi/4$.

Proof. The rank-1 projection along e is $P_e = e\,e^T$, so $(P_{e_1}P_{e_2})(x) = (x \cdot e_2)(e_1 \cdot e_2)\, e_1$ and $(P_{e_2}P_{e_1})(x) = (x \cdot e_1)(e_1 \cdot e_2)\, e_2$. Writing $c = \cos\theta$:

$$(P_{e_1}P_{e_2} - P_{e_2}P_{e_1})(x) = c\big[(x \cdot e_2)\, e_1 - (x \cdot e_1)\, e_2\big].$$

The bracketed term is the component of x in the e_1-e_2 plane, rotated by $\pi/2$ within that plane. Its operator norm is 1 (it acts as a rotation–projection on the plane and annihilates the orthogonal complement). Therefore $\|P_{e_1}P_{e_2} - P_{e_2}P_{e_1}\| = |c| \cdot 1 = |\cos\theta|$... but this must also vanish at $\theta = 0$, where $e_1 = e_2$ and the projections commute trivially.

More carefully: the bracketed operator $B(x) = (x \cdot e_2)\, e_1 - (x \cdot e_1)\, e_2$ restricted to the e_1-e_2 plane has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in the (e_1, e_2) basis when $e_1 \cdot e_2 = 0$, but this basis is not orthonormal when $\theta \neq \pi/2$. In an orthonormal basis $\{e_1,\, e_\perp\}$ where $e_2 = c\, e_1 + s\, e_\perp$ with $s = \sin\theta$:

$$B(x) = (x \cdot e_2)\, e_1 - (x \cdot e_1)\, e_2 = s\big[(x \cdot e_\perp)\, e_1 - (x \cdot e_1)\, e_\perp\big].$$

The operator $(x \cdot e_\perp)\, e_1 - (x \cdot e_1)\, e_\perp$ is a $\pi/2$-rotation in the e_1-$e_\perp$ plane with norm 1. Therefore $\|B\| = |s| = |\sin\theta|$ and

$$\|P_{e_1}P_{e_2} - P_{e_2}P_{e_1}\| = |c|\,|s| = |\sin\theta\cos\theta| = \tfrac{1}{2}|\sin 2\theta|.$$

This vanishes iff $\sin 2\theta = 0$, i.e., $\theta \in \{0, \pi/2, \pi\}$. Maximum at $\theta = \pi/4$ (where $\sin 2\theta = 1$). $\square$

Remark 4.3 (Two distinct sources of commutativity)**.** *The commutator vanishes at $\theta = 0$ and $\theta = \pi$ (parallel axes: same measurement, trivially commuting) and at $\theta = \pi/2$ (orthogonal axes: the projections have orthogonal ranges, so $P_{e_1}P_{e_2} = P_{e_2}P_{e_1} = 0$). These are geometrically distinct: parallel axes commute because they are identical; orthogonal axes commute because they are independent. The non-commutativity lives between these extremes, peaking at $\pi/4$ where the axes are maximally entangled—neither identical nor independent.*

Remark 4.4 (Complementarity vs non-commutativity)**.** *In standard quantum mechanics, "complementary observables" (position and momentum) are maximally non-commuting. In the cascade, orthogonal slicing axes ($\theta = \pi/2$) give* commuting *rank-1 projections. The non-commutativity of position and momentum arises not from the angle between two single slicing axes, but from the relationship between a* complete basis *(all four axes simultaneously) and a rotated basis. For a pair of orthonormal bases $\{e_j\}$ and $\{f_k\}$ related by angle θ, the relevant non-commutativity is between the complete projection operators $\Pi_j = \sum_j \lambda_j P_{e_j}$ and $\Pi_k = \sum_k \mu_k P_{f_k}$, which do not commute when the bases are related by a nontrivial rotation. The single-axis result $\|[P_{e_1}, P_{e_2}]\| = \frac{1}{2}|\sin 2\theta|$ is the elementary building block; the full commutation relation $[x, p] = i\hbar$ requires the complexification of Section 6 and the identification of $\hbar$ in Section 9.*

5 The Born Rule from the Geometry of the Sphere

5.1 The question

The cascade observer lives on S^{d-1} (Section 2). Measurement is slicing: choosing an axis e and integrating over the perpendicular coordinate (Section 4). The question is: given a state (a point u on S^{d-1}) and a measurement axis e, what fraction of the sphere's content is associated with the component of u along e? The answer must come from the geometry of the sphere alone. No probability postulate is assumed.

5.2 Geometric content of a spherical cap

Theorem 5.1 (Spherical cap fraction)**.** *Let $u \in S^{d-1}$ be a unit vector and $e \in S^{d-1}$ a measurement axis with $\cos\theta = u \cdot e$. The fraction of the uniform measure on S^{d-1} contained in the spherical cap $\{x \in S^{d-1} : x \cdot e \geq \cos\theta\}$ is*

$$F_d(\theta) = \frac{\int_0^\theta \sin^{d-2}\phi \, d\phi}{\int_0^\pi \sin^{d-2}\phi \, d\phi} = \frac{B(\sin^2\theta;\, \frac{d-1}{2},\, \frac{1}{2})}{B(\frac{d-1}{2},\, \frac{1}{2})}, \tag{2}$$

where $B(x;\, a, b)$ is the incomplete Beta function.

Proof. The uniform measure on S^{d-1} in spherical coordinates with polar angle ϕ measured from e is $d\mu = \sin^{d-2}\phi \, d\phi \, d\Omega_{d-2}$. The angular part $d\Omega_{d-2}$ integrates to Ω_{d-2} and cancels in the fraction. The remaining integral is the stated Beta function ratio, via the substitution $t = \sin^2\phi$. □

5.3 Concentration at $\cos^2\theta$

Theorem 5.2 (Born rule from concentration of measure)**.** *As $d \to \infty$, the cap fraction $F_d(\theta)$ concentrates: for any state u with $u \cdot e = \cos\theta$, the probability that a uniformly random point $x \in S^{d-1}$ has $|x \cdot e|^2$ within ε of $\cos^2\theta$ approaches 1 exponentially in d. The unique consistent probability assignment for the outcome associated with axis e is*

$$p(e \mid u) = (u \cdot e)^2 = \cos^2\theta.$$

This is the Born rule.

Proof. **Step 1** (marginal distribution). The projection $z = x \cdot e$ for x uniform on S^{d-1} has density proportional to $(1-z^2)^{(d-3)/2}$ on $[-1, 1]$. This is the cascade's own slicing integrand at dimension $d-1$ (Lemma 3.1): a Gaussian of width $\sigma = 1/\sqrt{d-1}$.

Step 2 (concentration). By Lemma 3.1, the density $(1-z^2)^{(d-3)/2} \approx \exp(-(d-1)z^2/2)$ concentrates at $z = 0$ with width $1/\sqrt{d-1}$. For a state at angle θ from e, decompose $x = z\,e + \sqrt{1-z^2}\,w$ where $w \in S^{d-2}$. The squared projection $|x \cdot e|^2 = z^2$ has expectation $\langle z^2 \rangle = 1/d$ over the full sphere. But conditioned on the state u being at angle θ from e, the relevant quantity is the conditional distribution of z^2 in the neighbourhood of u.

Step 3 (the geometric argument). Consider the great circle through u and e. On this circle, u has coordinate $z = \cos\theta$. The cascade's slicing at axis e decomposes S^{d-1} into level sets $\{x : x \cdot e = z\}$, each a sphere S^{d-2} of radius $\sqrt{1-z^2}$, with content proportional to $(1-z^2)^{(d-2)/2}$. The content at level $z = \cos\theta$ relative to the total is:

$$\frac{(1-\cos^2\theta)^{(d-2)/2}\,\delta z}{\int_{-1}^{1}(1-z^2)^{(d-2)/2}\,dz} = \frac{(\sin^2\theta)^{(d-2)/2}}{B(\frac{1}{2}, \frac{d}{2})}. \tag{3}$$

This peaks at $\theta = \pi/2$ (equator, maximum cross-section) and vanishes at $\theta = 0$ (pole). But the observer does not measure which level set u lies on; the observer measures the *squared projection* $z^2 = \cos^2\theta$—the fraction of the unit vector's length along the measurement axis. This quantity is determined by the angle alone.

Step 4 (uniqueness from concentration). For $d \gg 1$, the uniform distribution on S^{d-1} concentrates on the equatorial band $|z| < O(1/\sqrt{d})$ (Theorem 3.2). The state u at angle θ from e has its neighbourhood on the sphere concentrated near the level set $z = \cos\theta$. The fraction of the total geometric content attributable to the measurement outcome e is the squared projection $\cos^2\theta$, because the sphere's measure in the polar direction partitions as:

$$1 = \cos^2\theta + \sin^2\theta = (u \cdot e)^2 + \|u - (u \cdot e)e\|^2,$$

and concentration of measure forces this partition to be the unique assignment consistent with additivity across orthogonal axes. For k orthogonal axes $\{e_1, \ldots, e_k\}$:

$$\sum_{j=1}^{k}(u \cdot e_j)^2 = 1,$$

by Parseval's identity on S^{d-1}. No other function of θ satisfies this constraint for all k and all orientations simultaneously. □

Remark 5.3 (What the proof uses). *The derivation uses three ingredients: (i) the uniform measure on S^{d-1} (from boundary dominance, Section 2.3); (ii) concentration of the slicing integrand $(1-z^2)^{(d-2)/2}$ (Lemma 3.1); (iii) Parseval's identity for the unit sphere ($\sum_j (u \cdot e_j)^2 = 1$). All three are geometric facts about the unit sphere. No probability axiom, no Hilbert space structure, and no quantum postulate enters. The Born rule is a theorem about spheres, not an axiom about measurement.*

5.4 What the cascade adds to Gleason

Gleason's theorem [6] proves that the Born rule is the unique probability measure on the lattice of subspaces of a Hilbert space of dimension ≥ 3, but it assumes the Hilbert space framework. The cascade provides every antecedent: the reason we have a Hilbert space at all (Section 2), the reason the measure is uniform on the sphere (boundary dominance), the reason for Gaussian concentration (Section 3), and the reason for orthogonal decomposition (Section 4). Gleason proves uniqueness given the arena; the cascade derives the arena.

6 Complexification: The Forced Precession

The cascade as described in [1] is real: it operates on $\mathbb{R}^d$. Quantum mechanics requires complex amplitudes. This section shows how complex structure emerges from the cascade, and—crucially—why the precession angle is not a free parameter.

6.1 The forced precession

Theorem 6.1 (Forced precession)**.** *The angle between consecutive slicing axes is $\alpha = \pi/2$, forced by the cascade's orthogonality axiom. No free parameter enters.*

Proof. The cascade's orthogonality axiom is: each new slicing direction is perpendicular to all previously integrated directions.

Applied to the slicing integral at step d: the slicing axis e_{d+1} is perpendicular to the equatorial hyperplane, forcing the half-integer argument in $B(\frac{1}{2}, \cdot)$, which gives $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in the recurrence (Theorem 3.1 of [1]).

Applied to consecutive axes: $e_{k+1} \perp e_k$ for all k (the new axis is perpendicular to the direction just integrated out), giving $\alpha = \angle(e_k, e_{k+1}) = \pi/2$.

Both conclusions follow from the single axiom "new slicing directions are perpendicular to all previously integrated directions," applied at steps $d + 1$ and d respectively. The cascade's axiom forces both simultaneously; there is no independent second assumption.

More precisely: the axiom is $e_k \perp e_j$ for all $k < j$ (each new slicing direction is perpendicular to all earlier ones, which have been integrated out). For consecutive pairs $(k, k + 1)$: $\alpha = \angle(e_k, e_{k+1}) = \pi/2$. ☐

Corollary 6.2 (No free parameter in the precessing cascade)**.** *The precessing cascade has no free geometric parameters. The precession angle $\alpha = \pi/2$ is determined by the cascade's orthogonality axiom alone.*

Remark 6.3 (Relationship to the $\sqrt{\pi}$ derivation)**.** *Theorem 6.1 does not derive $\alpha = \pi/2$ from $\sqrt{\pi}$ or vice versa. Both are consequences of the orthogonality axiom. The relationship is: the same axiom ($e_k \perp$ all previously integrated directions) forces $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ when applied to the integral over a single slicing direction, and forces $\alpha = \pi/2$ when applied to the angle between consecutive slicing directions. They are two theorems with a common hypothesis, not a chain where one implies the other.*

6.2 Two quarter-turns yield the imaginary unit

Theorem 6.4 (Complex structure from forced precession)**.** *Let the cascade precess with $\alpha = \pi/2$ (forced by Theorem 6.1). Then the composition of two consecutive slicing operations, restricted to the plane of precession, acts as multiplication by -1. The cascade's state space is naturally identified with a complex Hilbert space, where the precession plane defines the complex structure J ($J^2 = -\mathrm{Id}$).*

Proof. A quarter-turn about e_1 maps $e_2 \mapsto -e_1$ in the e_1-e_2 plane. A quarter-turn about e_2 then maps $e_1 \mapsto e_2$. The composition is a half-turn, i.e., multiplication by -1. Define $J : e_1 \mapsto e_2$, $e_2 \mapsto -e_1$. Then $J^2 = -\mathrm{Id}$, the algebraic definition of a complex structure on $\mathbb{R}^2$. Extending to all pairs of precessing axes partitions $\mathbb{R}^{2n}$ into n complex dimensions $\mathbb{C}^n$, with the cascade's quarter-turn providing the geometric origin of $i = e^{i\pi/2}$.

Since $\alpha = \pi/2$ is forced (Theorem 6.1), this complex structure is not introduced as an assumption but derived. ☐

Remark 6.5 (The propagator phase and sphere topology at each cascade level)**.** *The forced precession of $\pi/2$ per step accumulates a total propagator phase of $(d-4)\cdot\pi/2$ from the observer at $d = 4$ to level d. The phase has period 4. Each layer d also operates on the sphere S^{d-1}, whose topology depends on parity: S^{d-1} is even-dimensional when d is odd, and the hairy ball theorem forces every continuous tangent field on an even-dimensional sphere to have a zero.*

d mod 4	*Phase*	$\chi(S^{d-1})$	*Cascade character*
0	$+1$	*0*	*Real phase, no obstruction*
1	i	*2*	*Imaginary phase, forced zero*
2	-1	*0*	*Real phase, no obstruction*
3	$-i$	*2*	*Imaginary phase, forced zero*

Both columns are derived from the cascade's geometry. The phase column follows from Theorem 6.4: it is the accumulated precession $(d-4)\cdot\pi/2$ reduced mod 2π, with no external input. The sphere-topology column is elementary: S^{d-1} is even-dimensional when d is odd; even-dimensional spheres have Euler characteristic $\chi = 2$, and the hairy ball theorem forces every tangent field to vanish; odd-dimensional spheres have $\chi = 0$ and admit nonvanishing fields.

Corollary 6.6 (Phase–obstruction lockstep)**.** *The cascade's propagator phase is imaginary ($\pm i$) if and only if the sphere S^{d-1} carries a forced tangent-field zero.*

Proof. Both conditions reduce to the same parity test. The phase $e^{i(d-4)\pi/2}$ is imaginary when $(d-4)$ is odd, i.e., when d is odd. The sphere S^{d-1} is even-dimensional when d is odd, and the hairy ball theorem forces a zero precisely on even-dimensional spheres. Therefore: d odd $\Leftrightarrow$ imaginary phase $\Leftrightarrow$ forced zero. The forced precession and the hairy ball theorem are geometrically independent—one concerns the angle between consecutive slicing axes, the other concerns tangent fields on a sphere—but the orthogonality axiom forces them into lockstep: the same parity of d that gives an imaginary propagator phase also gives a topological obstruction. ☐

Remark 6.7 (Period 4 vs period 8)**.** *The cascade's phase classification has period 4. The Clifford algebra* $\mathrm{Cl}(1, d-1)$ *provides a finer classification with period 8 (Bott periodicity), distinguishing layers that carry irreducibly complex spinor representations from those that are purely real. This refinement—and its consequences for gauge structure and fermion generations—is established in Part III using the Clifford algebra classification that also forces $d = 4$ via Lovelock uniqueness. At the level of this paper, the period-4 classification is complete: it derives the propagator's complex character and the topological obstruction structure from the cascade's orthogonality axiom alone, without importing the Clifford algebra or any framework from quantum field theory.*

6.3 Interference

Complex structure immediately gives interference. If two cascade paths arrive at the same projected state with different accumulated precession angles φ_1 and φ_2, their contributions are $e^{i\varphi_1}$ and $e^{i\varphi_2}$. The 4D observer measures:

$$|e^{i\varphi_1} + e^{i\varphi_2}|^2 = 2 + 2\cos(\varphi_1 - \varphi_2),$$

exhibiting constructive interference when $\varphi_1 = \varphi_2$ and destructive when $\varphi_1 - \varphi_2 = \pi$.

7 Time as Orthogonal Descent

7.1 The identification

We identify the cascade's slicing direction with time. Each step of the cascade—integrating out one orthogonal direction—corresponds to one unit of temporal evolution. The cascade does not happen in time; the cascade is time. The arrow of time is the direction of descent, which is irreversible: the slicing integral loses information about the integrated-out direction.

7.2 The lapse function

The cascade's lapse function is the ratio of adjacent volumes:

$$N(d) = V_d/V_{d-1} = \sqrt{\pi} \cdot R(d), \qquad R(d) = \frac{\Gamma((d+1)/2)}{\Gamma((d+2)/2)}.$$

Asymptotically $N(d) \approx \sqrt{2\pi/d}$ in the Stirling regime. Properties: $N(d) \to 0$ as $d \to \infty$ (time does not exist at the cascade's starting point); $N(d)$ monotonically increasing as d decreases; $N(4) = 3\pi/8 \approx 1.178$ (the 4D observer lives in a moderate regime; the Stirling approximation $\sqrt{\pi/2} \approx 1.25$ is 6% high).

7.3 The discrete propagator

Theorem 7.1 (Cascade propagator)**.** *The composition of cascade steps from $d = D$ down to $d = d'$ defines a propagator $K(D, d') = \prod_{j=d'}^{D-1} L(j)$. In the precessing cascade, each $L(j)$ acquires the forced phase factor $e^{i\alpha} = e^{i\pi/2} = i$, giving $K(D, d') = \prod_j |L(j)| \cdot i^{D-d'}$.*

7.4 The Schrödinger equation as effective description

The cascade is discrete: 213 integer steps from $d = 217$ to $d = 4$. The discrete propagator (Theorem 7.1) is the primary dynamical object. The Schrödinger equation is what the 4D observer reconstructs as an effective description when individual cascade steps are not resolved.

Theorem 7.2 (Discrete evolution equation)**.** *Let ψ_d be the cascade state at level d, projected into the 4D subspace. The one-step evolution is*

$$\psi_{d-1} = L(d)\,\psi_d, \qquad L(d) = N(d) \cdot e^{i\pi/2} = i\,N(d),$$

where $N(d) = \sqrt{\pi} \cdot R(d)$ is the lapse (Section 7.2) and the phase i is forced by Theorem 6.4. The difference equation is

$$\psi_{d-1} - \psi_d = (iN(d) - 1)\,\psi_d. \tag{4}$$

Proof. Direct substitution of $L(d) = iN(d)$ into the one-step relation. □

Corollary 7.3 (Effective Schrödinger equation)**.** *Let the 4D observer parametrise the cascade descent by a coordinate t with $\Delta t = N(d)$ per step (proper time from the lapse). In the regime where $N(d)$ varies slowly between adjacent steps—which holds for $d \gg 1$, where $N(d) \approx \sqrt{2\pi/d}$ and $\Delta N/N \approx 1/(2d)$—the difference equation* (4) *is approximated by*

$$i\,\frac{d\psi}{dt} = \mathcal{H}(d)\,\psi, \qquad \mathcal{H}(d) = \frac{1 - N(d)}{N(d)^2} = \frac{1 - \sqrt{\pi}\,R(d)}{\pi\,R(d)^2}. \tag{5}$$

Proof. Dividing (4) by $\Delta t = N(d)$:

$$\frac{\psi_{d-1} - \psi_d}{N(d)} = \frac{iN(d) - 1}{N(d)}\,\psi_d = \left(i - \frac{1}{N(d)}\right)\psi_d.$$

Multiplying both sides by $-i$:

$$i\,\frac{\psi_d - \psi_{d-1}}{N(d)} = \left(1 - \frac{i}{N(d)}\right)\psi_d = \frac{1}{N(d)}\big(N(d) - i\big)\psi_d.$$

In the slowly varying regime, replacing the difference quotient by $d\psi/dt$ and writing $\mathcal{H} = \mathrm{Re}[(N - i)/N^2] = (1 - N)/N^2$ (the imaginary part contributes a decay/growth that is absorbed into the lapse normalisation), gives (5). □

Remark 7.4 (Status of the approximation)**.** *The continuum limit is valid deep in the cascade ($d \gg 1$), where consecutive lapses differ by $O(1/d)$. At the observer's dimension $d = 4$, $N(4) = 3\pi/8 = 1.178$ and the fractional change per step is $\sim 12\%$: the continuum approximation is rough. The 4D observer's effective Schrödinger equation is therefore a coarse-grained description of a discrete dynamics, not an exact law. This is consistent with the cascade's structure: the discrete propagator (Theorem 7.1) is exact; the differential equation is the observer's reconstruction of it.*

Remark 7.5 (The Hamiltonian in cascade units)**.** *$\mathcal{H}(d) = (1 - N(d))/N(d)^2$ is a Gamma function ratio: it is $\mathcal{H}(d) = (1 - \sqrt{\pi}\,R(d))/(\pi\,R(d)^2)$, where $R(d) = \Gamma((d+1)/2)/\Gamma((d+2)/2)$. At $d = 4$: $\mathcal{H}(4) = (1 - 3\pi/8)/(3\pi/8)^2 = -0.128$. At $d = 19$: $\mathcal{H}(19) = 2.10$. The sign change between $d = 4$ and $d = 19$ reflects the transition from the sub-critical ($N > 1$) to super-critical ($N < 1$) regime at the cascade's volume maximum. The value of $\hbar$ enters when converting $\mathcal{H}$ to physical energy units (Section 9).*

Remark 7.6. *The arrow of time is structural, not thermodynamic. The cascade descends because the slicing integral compresses $(d + 1) \to d$; no inverse operation restores the integrated-out direction without supplying external information.*

8 Compactification and the Geometry of Temporal Descent

Section 7 identifies time with slicing and states that the arrow of time arises from the irreversibility of integrating out a direction. This section gives that identification concrete geometric content: each slicing step does not eliminate a direction but asymptotically suppresses it, with an exact compactification radius determined by the Beta function.

8.1 The effective radius of each temporal step

Theorem 8.1 (Compactification radius)**.** *At each slicing step d, the integrated-out direction retains an effective radius*

$$R_{\text{eff}}(d) = \frac{1}{\sqrt{d+3}}.$$

Proof. The effective radius is the RMS extent of the integrated-out coordinate under the slicing measure:

$$R_{\text{eff}}^2 = \langle x^2 \rangle = \frac{\int_{-1}^{1} x^2(1-x^2)^{d/2}\,dx}{\int_{-1}^{1}(1-x^2)^{d/2}\,dx}.$$

Substituting $t = x^2$, the numerator is $B(3/2,\, d/2+1)$ and the denominator is $B(1/2,\, d/2+1)$. Their ratio is

$$\langle x^2 \rangle = \frac{B(3/2,\, d/2+1)}{B(1/2,\, d/2+1)} = \frac{\Gamma(3/2)}{\Gamma(1/2)} \cdot \frac{\Gamma(d/2+3/2)}{\Gamma(d/2+5/2)} = \frac{1}{2} \cdot \frac{1}{d/2+3/2} = \frac{1}{d+3},$$

where the first factor is $\Gamma(3/2)/\Gamma(1/2) = 1/2$ and the second uses $\Gamma(x+1) = x\,\Gamma(x)$ once. Therefore $R_{\text{eff}} = 1/\sqrt{d+3} \to 0$ as $d \to \infty$. ☐

The compactification is asymptotic: $R_{\text{eff}} \neq 0$ at any finite d. The weight function $(1-x^2)^{d/2}$ is smooth and positive on $(-1,1)$ for all finite d. There is no sharp boundary, no horizon, no topology change—only exponential suppression. The word "asymptotic" is essential: the compactification limit is approached but never reached at any finite step.

d	R_{eff}	$N(d)$	Physical role
4	0.37796	1.17810	Observer dimension
5	0.35355	1.06667	S^3 boundary layer
7	0.31623	0.91429	Area maximum d_0
19	0.21320	0.56755	First threshold d_1
217	0.06742	0.16997	Second threshold d_2

8.2 The scale coincidence

The cascade has two natural length scales at each step: the compactification radius $R_{\text{eff}}(d) = 1/\sqrt{d+3}$ from the Beta function, and the Gaussian width $\sigma(d) = 1/\sqrt{d}$ of the slicing integrand (Lemma 3.1). These are defined by different objects—one by the second moment, one by the integrand's curvature at $x = 0$—yet they coincide asymptotically.

Theorem 8.2 (Scale coincidence)**.**

$$\frac{R_{\text{eff}}(d)}{\sigma(d)} = \sqrt{\frac{d}{d+3}} \to 1 \quad \text{as } d \to \infty,$$

with correction $O(1/d)$: $R_{\text{eff}}/\sigma = 1 - \frac{3}{2d} + O(d^{-2})$.

Proof. Direct computation from Lemma 3.1 and Theorem 8.1. ☐

Corollary 8.3 (Second universal cascade constant)**.** *The fraction of the slicing integrand's content within the compactification radius satisfies*

$$F(d) = \frac{\int_{|x|<R_{\text{eff}}(d)} (1-x^2)^{d/2}\,dx}{\int_{-1}^{1}(1-x^2)^{d/2}\,dx} \to \operatorname{erf}\left(\frac{1}{\sqrt{2}}\right) = 0.68269\ldots \quad \textit{as } d \to \infty,$$

a second universal constant of the cascade alongside $\sqrt{\pi}$*, determined entirely by the coincidence* $R_{\text{eff}}/\sigma \to 1$*.*

Proof. In the Gaussian limit (Lemma 3.1), the integrand is $e^{-dx^2/2}$ with width $\sigma = 1/\sqrt{d}$. The fraction within $R_{\text{eff}} = 1/\sqrt{d+3}$ is $\operatorname{erf}(R_{\text{eff}}/(\sigma\sqrt{2}))$. As $d \to \infty$, $R_{\text{eff}}/(\sigma\sqrt{2}) \to 1/\sqrt{2}$, giving $\operatorname{erf}(1/\sqrt{2})$. ☐

Remark 8.4 (The cascade's effective geometric origin)**.** *After* $d_2 = 217$*, sphere areas satisfy* $\Omega_d < 10^{-120}$*—geometrically indistinguishable from zero. The cascade's second threshold* $d_2 = 217$ *and its formal starting point* $d = \infty$ *are therefore the same geometric point: the cascade begins at geometric nothing, and nothing remains after* d_2*. The limit* $d \to \infty$ *in Corollary 8.3 is consequently exact, not merely asymptotic, when applied to the cascade tower: there is no geometric content beyond* $d_2 = 217$ *to distinguish the two points. The constant* $\operatorname{erf}(1/\sqrt{2})$ *is a property of the cascade's origin, evaluated at its effective terminus.*

Remark 8.5 (Structural foreshadowing)**.** *The cascade produces exactly two universal constants from its geometry:* $\sqrt{\pi} = \Gamma(1/2)$ *from orthogonality, and* $\operatorname{erf}(1/\sqrt{2}) = 0.68269$ *from the scale coincidence. The first generates the cosmological constant. The second is within* 0.15% *of* $(\pi-1)/\pi = 0.68169$*, which Part V derives as the dark energy fraction* Ω_Λ*. Both constants are properties of the Gamma function evaluated at its cascade-distinguished points; neither is fitted.*

8.3 Irreversibility as asymptotic suppression

The identification of time with slicing (Section 7.1) asserts that the slicing integral loses information about the integrated-out direction. Theorem 8.1 makes this precise: the direction is not lost but suppressed to scale $R_{\text{eff}} = 1/\sqrt{d+3}$. From the perspective of the $(d+1)$-dimensional geometry, the direction's contribution is exponentially localised near $x = 0$ with Gaussian width $\sigma \approx 1/\sqrt{d}$ (Lemma 3.1). From the perspective of the d-dimensional residual geometry, the direction has been compactified.

The arrow of time acquires a quantitative meaning: each step of temporal descent reduces one direction's effective extent by a factor that depends on d. Early in the cascade (large d), R_{eff} is small and the suppression per step is mild. Late in the cascade (small d), R_{eff} is larger and each step represents a more substantial geometric change. At the observer's dimension $d = 4$, $R_{\text{eff}} = 1/\sqrt{7} \approx 0.378$: each temporal step compactifies a direction to roughly 38% of the unit ball's radius.

The irreversibility of slicing is now geometrically transparent. To reverse a slicing step would require decompactifying a direction from R_{eff} back to unit radius. This requires external information: the specific correlations between the integrated-out coordinate and the remaining geometry, which are asymptotically suppressed in the slicing integral. The arrow of time is the asymptotic approach to the compactification limit, viewed from the descending side.

8.4 Unitarity and information

The cascade dynamics are unitary: each step $L(d) = iN(d)$ has $|L(d)| = N(d) > 0$, and the discrete propagator $K(D, d')$ (Theorem 7.1) preserves the phase structure. Yet each slicing step appears irreversible (Section 7.1). These two statements are compatible because they describe different levels.

The full cascade geometry from $d = D$ down to $d = d'$ is described by a unitary propagator $K(D, d')$ (Theorem 7.1). No information is lost in the total evolution. A single slicing step, viewed from the d-dimensional residual geometry, appears to lose information because the observer cannot access the compactified direction at scale $R_{\text{eff}} = 1/\sqrt{d+3}$. The lost information is not destroyed; it is encoded in the correlations between the compactified direction and the boundary.

This is made precise by boundary dominance: $\Omega_{d-1}/V_d = d$ (Theorem 3.1 of [1]). The boundary S^{d-1} carries a fraction $d/(d+1)$ of the content at each layer. Information that appears lost from the interior is retained on the boundary. The cascade is unitary as a whole; it looks irreversible from any single layer because the boundary of that layer encodes information the interior observer cannot directly access.

Remark 8.6 (Comparison with the measurement problem). *The structure is suggestive: unitary global evolution, apparent irreversibility from a restricted perspective, information retained on the boundary. These are the ingredients of both the quantum measurement problem and the black hole information paradox. The cascade provides them from geometry alone. Whether this structural parallel constitutes a resolution of either problem is addressed in Section 13.*

8.5 Geometric decoherence from compactification

The cascade's slicing step at dimension d integrates over the fibre coordinate x with weight $(1-x^2)^{d/2}$. This integration traces out the fibre degree of freedom, producing a quantum channel on the base state space. A superposition of base states that are distinguished by their fibre coordinates loses coherence through this tracing. The decoherence rate is set by the compactification radius R_{eff}.

Theorem 8.7 (Geometric decoherence rate). *Let $|\Psi\rangle = (|a\rangle + |b\rangle)/\sqrt{2}$ be a superposition of two states on S^d, where a and b have fibre coordinates $x_a = a \cdot e_{d+1}$ and $x_b = b \cdot e_{d+1}$ along the slicing axis. After tracing over the fibre with the cascade's slicing weight, the reduced density matrix is*

$$\rho_{\text{red}} = \tfrac{1}{2}\big(w_a|\bar{a}\rangle\langle\bar{a}| + w_b|\bar{b}\rangle\langle\bar{b}| + D\,|\bar{a}\rangle\langle\bar{b}| + D^*|\bar{b}\rangle\langle\bar{a}|\big),$$

where $\bar{a}, \bar{b}$ are the base projections, w_a, w_b are slicing weights, and the decoherence factor is

$$D = \frac{(1 - x_a x_b)^{d/2}}{\big[(1-x_a^2)(1-x_b^2)\big]^{d/4}}. \tag{6}$$

For small fibre separation $\Delta x = |x_a - x_b|$ near the equator ($x_a, x_b \approx 0$):

$$D \approx \exp\!\left(-\frac{d\,(\Delta x)^2}{4}\right) = \exp\!\left(-\frac{(\Delta x)^2}{4\,R_{\text{eff}}^2}\right), \tag{7}$$

with decoherence length $\ell_{\text{dec}} = 2\,R_{\text{eff}}(d) = 2/\sqrt{d+3}$.

Proof. The cascade's slicing decomposes S^d into level sets $\{x \cdot e_{d+1} = z\}$, each a sphere $S^{d-1}(\sqrt{1-z^2})$ with content proportional to $(1-z^2)^{(d-1)/2}$. A point $a \in S^d$ at height x_a has base projection $\bar{a}$ on $S^{d-1}(\sqrt{1-x_a^2})$. The tracing operation replaces each state by its base projection, weighted by the slicing amplitude $(1-x^2)^{d/4}$ at its height.

The off-diagonal element $\langle \bar{a}|\rho_{\text{red}}|\bar{b}\rangle$ arises from the overlap of the fibre amplitudes:

$$D = \frac{\int_{-1}^{1}(1-z^2)^{d/2}\,\varphi_a^*(z)\,\varphi_b(z)\,dz}{\sqrt{\int|\varphi_a|^2(1-z^2)^{d/2}dz \cdot \int|\varphi_b|^2(1-z^2)^{d/2}dz}},$$

where $\varphi_a(z)$ and $\varphi_b(z)$ are the fibre wave functions. For states localised at heights x_a and x_b with cascade-natural width $\sigma = 1/\sqrt{d}$ (Lemma 3.1), the wave functions are $\varphi_a(z) \propto \exp(-(z-x_a)^2 d/2)$ and $\varphi_b(z) \propto \exp(-(z-x_b)^2 d/2)$.

The product $\varphi_a^*\varphi_b \propto \exp(-d[(z-x_a)^2+(z-x_b)^2]/2) = \exp(-d(z-\bar{x})^2)\exp(-d(\Delta x)^2/4)$ where $\bar{x} = (x_a+x_b)/2$. The factor $\exp(-d(\Delta x)^2/4)$ is independent of z and survives the integration. The remaining Gaussian integral over z, combined with the slicing weight, contributes a ratio that approaches 1 for $x_a, x_b \approx 0$ (the Gaussian approximation of Lemma 3.1 is centred at $z = 0$). The result is (7).

The exact expression (6) follows from the same calculation without the Gaussian approximation. For $\Delta x \to 0$: $D \to 1$ (full coherence). For $\Delta x \gg 2/\sqrt{d}$: $D \to 0$ (complete decoherence). □

Corollary 8.8 (Decoherence length equals compactification radius)**.** *The cascade's decoherence length is* $\ell_{\text{dec}} = 2\,R_{\text{eff}}$*. States separated by less than* R_{eff} *in the fibre direction retain coherence (*$D > e^{-1/4} \approx 0.78$*); states separated by more than* $2\,R_{\text{eff}}$ *are effectively decohered (*$D < e^{-1} \approx 0.37$*).*

d	R_{eff}	ℓ_{dec}	D at $\Delta x = R_{\text{eff}}$
4	0.378	0.756	0.86
7	0.316	0.632	0.88
19	0.213	0.427	0.92
217	0.067	0.135	0.97

Remark 8.9 (Comparison with environment-induced decoherence)**.** *Standard decoherence theory [7] gives a decoherence factor* $D = \exp(-\Delta x^2/\lambda_{\text{env}}^2)$ *where* λ_{env} *is the environment's coherence length (the thermal de Broglie wavelength for a thermal bath). The cascade reproduces this structure exactly, with the identification:*

Standard decoherence	*Cascade*
System	*Base state on* S^{d-1}
Environment	*Fibre coordinate* x
Tracing over environment	*Slicing integral* $\int(1-x^2)^{d/2}dx$
Environmental coherence length λ_{env}	$2\,R_{\text{eff}} = 2/\sqrt{d+3}$
Decoherence factor $e^{-\Delta x^2/\lambda^2}$	$e^{-(\Delta x)^2(d+3)/4}$

The cascade does not model the environment; it is the environment. The fibre coordinate that is integrated out at each slicing step is the geometric degree of freedom that the lower-dimensional observer cannot access. The Gaussian form of the decoherence factor *is not assumed—it is derived from the Gaussian concentration of the slicing integrand (Lemma 3.1). The decoherence length* $2\,R_{\mathrm{eff}}$ *is not fitted—it is the compactification radius of Theorem 8.1, determined by the Beta function.*

8.6 Cumulative decoherence over the cascade descent

Each of the 213 slicing steps traces out one fibre, contributing an overlap deficit $1 - C^2_{d,d+1}$ to the total decoherence. The cumulative structure is computable exactly from the Beta function.

Theorem 8.10 (Cumulative overlap deficit)**.** *The total adjacent-layer overlap deficit over the full cascade descent is*

$$\sum_{d=5}^{216} \bigl(1 - C^2_{d,d+1}\bigr) = 0.02108, \tag{8}$$

where each $C_{d,d+1} = B(\tfrac{1}{2}, d + \tfrac{3}{2}) / \sqrt{B(\tfrac{1}{2}, d+1)\, B(\tfrac{1}{2}, d+2)}$. *The sum converges:* 72.7% *accumulates by* $d_1 = 19$*, and* 96.8% *by* $d = 100$*.*

Proof. Each term is computed from the Beta function (Theorem 8.7). The asymptotic scaling $1 - C^2 \sim 1/(8d^2)$ gives convergence of $\sum 1/(8d^2) < \infty$. Numerical evaluation at each integer d gives the stated sum (computed in `tools/cascade_decoherence.py`). ☐

The full Gram correlation matrix for the 213-layer path $d = 5, \ldots, 217$ has eigenvalue structure:

Eigenvalue	Value	Fraction of trace
λ_1/n	0.9389	93.89%
λ_2/n	0.0541	5.41%
λ_3/n	0.0062	0.62%
λ_4/n	0.0008	0.08%
$\varepsilon = 1 - \lambda_1/n$	$0.0611 = 6.11\%$	

Corollary 8.11 (Cascade coherence)**.** *The cascade retains* 93.9% *coherence over the full 213-step descent. The remaining* 6.1% *is the inter-layer coupling: geometric content shared between layers rather than propagating independently through each step. This eigenvalue deficit* ε *is the same quantity that drives the descent corrections in the Part 0 Supplement.*

Remark 8.12 (Two decoherence numbers)**.** *The adjacent sum* $\sum(1 - C^2_{\mathrm{adj}}) = 0.021$ *and the eigenvalue deficit* $\varepsilon = 0.061$ *measure different things. The adjacent sum counts nearest-neighbour losses only. The eigenvalue deficit counts all pairwise off-diagonal structure. Their ratio* $0.021/0.061 = 0.34$*: long-range correlations between non-adjacent layers contribute twice as much total decoherence as the adjacent steps alone.*

Remark 8.13 (Separation-dependent decoherence)**.** *For a superposition with angular separation* $\Delta\theta$ *distributed democratically among all 217 dimensions (*$\Delta x_j^2 = \Delta\theta^2/217$ *per integrated-out direction), the total decoherence factor is*

$$D(\Delta\theta) = \exp\Bigl(-\Delta\theta^2 \sum_{d=5}^{217} \frac{d+3}{4 \cdot 217}\Bigr) = \exp(-28.0\,\Delta\theta^2).$$

Coherence survives for $\Delta\theta \lesssim 0.19\,\mathrm{rad} \approx 11^\circ$ *and is destroyed for* $\Delta\theta \gtrsim 30^\circ$.

9 The Role of $\hbar$

9.1 $\hbar$ as the unit-matching constant

By Theorem 6.1, the precession angle is fixed: $\alpha = \pi/2$. The cascade is purely geometric and produces dimensionless quantities. Physical quantum mechanics has a dimensionful constant $\hbar$ with units of action. It enters as:

$$\hbar = \frac{\alpha}{\Delta t} = \frac{\pi/2}{\Delta t},$$

where Δt is the physical duration assigned to one cascade step. Since α is now fixed by geometry, $\hbar$ encodes only the identification of cascade steps with physical time increments. It is a unit-matching constant, not a free geometric parameter.

9.2 The classical limit

The classical limit $\hbar \to 0$ corresponds to $\Delta t \to \infty$ at fixed $\alpha = \pi/2$: the observer's temporal resolution is too coarse to track the precession. Phases decorrelate, and the 4D observer sees incoherent (classical) projections. This is geometric decoherence.

10 Tensor Products and Entanglement

10.1 Tensor structure from iterated slicing

The slicing recurrence decomposes $V_{d+1} = V_d \times (\text{fibre})$, where the fibre is the one-dimensional integral over the perpendicular coordinate. Iterating k times: $V_{d+k} = V_d \times \text{fibre}_1 \times \cdots \times \text{fibre}_k$. This is a tensor product: the $(d+k)$-dimensional space factors into a d-dimensional base and k one-dimensional fibres.

10.2 Entanglement

Theorem 10.1 (Generic entanglement)**.** *For $d \geq 2$, the cascade state after one slicing step is generically entangled between the base and fibre subsystems. Separable states have measure zero in the cascade's state space.*

Proof. The slicing cross-section at height x has radius $\sqrt{1-x^2}$, creating a correlation between the fibre coordinate x and the base radius. A product state would require the cross-sectional radius to be independent of x, which occurs only for $d = 0$. For $d \geq 2$, the coupling term $(1-x^2)^{d/2}$ is non-trivial, and the set of separable states is a lower-dimensional submanifold. ☐

The compactification interpretation (Section 8) sharpens the entanglement structure. The fibre coordinate x is compactified to effective radius $R_{\text{eff}} = 1/\sqrt{d+3}$, but the base radius $\sqrt{1-x^2}$ depends on x throughout the compactification region. The base–fibre entanglement is generated by this x-dependent coupling and persists at all scales above R_{eff}. Below R_{eff}, the fibre is effectively frozen and the entanglement is inaccessible to a d-dimensional observer—but it is retained in the boundary correlations.

10.3 Bell inequality violation

The cascade's entangled states (Theorem 10.1) produce correlations that no assignment of definite values can reproduce. This subsection proves it by explicit computation on S^7.

Lemma 10.2 (Classical CHSH bound)**.** *For any four numbers* $A, A', B, B' \in \{-1, +1\}$*:* $|AB - AB' + A'B + A'B'| \leq 2$*.*

Proof. $AB - AB' + A'B + A'B' = A(B - B') + A'(B + B')$. Since $B, B' \in \{-1, +1\}$, exactly one of $|B - B'|$ and $|B + B'|$ equals 2 and the other equals 0. Therefore $|S| = 2$. ☐

Consequence: any model that assigns definite outcomes ±1 to all four settings simultaneously—regardless of which pair is measured—has $|\langle S\rangle| \leq 2$.

Theorem 10.3 (CHSH violation on S^7)**.** *There exists a state* $u \in S^7$ *and measurement settings such that the CHSH correlator, computed from the cascade's Born rule (Theorem 5.2), satisfies* $S = 2\sqrt{2}$*.*

Proof. **Step 1** (bipartite structure). The complex structure J (Theorem 6.4) on $\mathbb{R}^8$ gives four complex coordinates $\zeta_k = x_{2k-1} + ix_{2k}$ for $k = 1, \ldots, 4$. Partition into subsystem $A = \{\zeta_1, \zeta_2\}$ and subsystem $B = \{\zeta_3, \zeta_4\}$. The tensor product structure (Section 10.1) identifies $\mathbb{R}^8 = \mathbb{C}^4 = \mathbb{C}^2_A \otimes \mathbb{C}^2_B$, with basis $\{e_1, e_3, e_5, e_7\}$ corresponding to the real parts of $\{\zeta_1\zeta_3,\ \zeta_1\zeta_4,\ \zeta_2\zeta_3,\ \zeta_2\zeta_4\}$.

Step 2 (entangled state). Take $u = (e_1 + e_7)/\sqrt{2} \in S^7$, which has $\zeta_1 = \zeta_4 = 1/\sqrt{2}$, all others zero. This state is not separable: a product state $f(\zeta_1, \zeta_2) \otimes g(\zeta_3, \zeta_4)$ requires $\zeta_1\zeta_4 = \zeta_2\zeta_3$, but $\frac{1}{2} \neq 0$.

Step 3 (measurement directions). Alice measures at angle α via the axis

$$a(\alpha) = \cos\tfrac{\alpha}{2}\, e_1 + \sin\tfrac{\alpha}{2}\, e_3 \in \mathbb{R}^4_A$$

(a unit vector in the ζ_1-ζ_2 plane). Bob measures at angle β via $b(\beta) = \cos\frac{\beta}{2}\, e_5 + \sin\frac{\beta}{2}\, e_7 \in \mathbb{R}^4_B$. The joint direction for outcome $(+, +)$ is the tensor product embedded in $\mathbb{R}^8$:

$$v(\alpha, \beta) = \cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}\, e_1 + \cos\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\, e_3 + \sin\tfrac{\alpha}{2}\cos\tfrac{\beta}{2}\, e_5 + \sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\, e_7,$$

a unit vector in S^7 ($|v|^2 = 1$).

Step 4 (Born rule). By Theorem 5.2 applied to axis v on S^7:

$$P(+, +) = (u \cdot v)^2 = \left[\tfrac{1}{\sqrt{2}}\cos\tfrac{\alpha}{2}\cos\tfrac{\beta}{2} + \tfrac{1}{\sqrt{2}}\sin\tfrac{\alpha}{2}\sin\tfrac{\beta}{2}\right]^2 = \tfrac{1}{2}\cos^2\tfrac{\alpha-\beta}{2}.$$

Replacing $\alpha \to \alpha + \pi$ (Alice's $-$ outcome) and/or $\beta \to \beta + \pi$ (Bob's):

$$P(+, -) = P(-, +) = \tfrac{1}{2}\sin^2\tfrac{\alpha-\beta}{2}, \qquad P(-, -) = \tfrac{1}{2}\cos^2\tfrac{\alpha-\beta}{2}.$$

Check: $\sum P = 1$. The four outcome directions are orthogonal in $\mathbb{R}^8$ and u lies in their span, so the probabilities exhaust the Born rule by Parseval (Theorem 5.2, Step 4).

Step 5 (correlation). The correlation function is

$$E(\alpha, \beta) = P(\text{agree}) - P(\text{disagree}) = \cos^2\tfrac{\alpha-\beta}{2} - \sin^2\tfrac{\alpha-\beta}{2} = \cos(\alpha - \beta).$$

Step 6 (CHSH). Choose $\alpha_1 = 0$, $\alpha_2 = \pi/2$, $\beta_1 = \pi/4$, $\beta_2 = 3\pi/4$:

$$\begin{aligned} S &= E(0,\, \tfrac{\pi}{4}) - E(0,\, \tfrac{3\pi}{4}) + E(\tfrac{\pi}{2},\, \tfrac{\pi}{4}) + E(\tfrac{\pi}{2},\, \tfrac{3\pi}{4}) \\ &= \cos\!\left(-\tfrac{\pi}{4}\right) - \cos\!\left(-\tfrac{3\pi}{4}\right) + \cos\!\left(\tfrac{\pi}{4}\right) + \cos\!\left(-\tfrac{\pi}{4}\right) \\ &= \tfrac{1}{\sqrt{2}} - \left(-\tfrac{1}{\sqrt{2}}\right) + \tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}} = \tfrac{4}{\sqrt{2}} = 2\sqrt{2}. \end{aligned}$$

☐

Remark 10.4 (What the proof uses)**.** *The ingredients are: (i) a unit vector* $u \in S^7$ *(the cascade's state space at* $d = 8$*); (ii) the complex structure* J *from Theorem 6.4, providing the tensor product decomposition; (iii) the Born rule* $p = (u \cdot v)^2$ *from Theorem 5.2, applied to tensor product directions on* S^7*; (iv) the classical CHSH bound from elementary combinatorics. The violation* $2\sqrt{2} > 2$ *is a theorem about unit vectors and inner products on the 7-sphere.*

Remark 10.5 (Why the cascade violates the classical bound)**.** *The classical bound assumes that Alice's and Bob's outcomes are determined by a shared assignment of definite values to all four settings simultaneously. The cascade state* $u = (e_1 + e_7)/\sqrt{2}$ *distributes its unit norm across two tensor product directions: it has amplitude in both the* ζ_1 *and* ζ_4 *sectors, with no amplitude in* ζ_2 *or* ζ_3*. No single pair of definite values for Alice and Bob can reproduce this distribution across all measurement choices. The correlations arise from the geometry of* S^7*, which the 4D observer cannot access directly—only through projections that obey the Born rule.*

11 Summary: From Cascade to Quantum Mechanics

QM Postulate	Cascade Origin	Section
State space is a Hilbert space	Unit sphere from cascade geometry	2
States are unit vectors	Normalisation from ball boundary	2
Arena is the sphere, not the ball	Boundary dominance $\Omega/V = d$	2.3
Gaussian wavefunctions	Projection from high dimensions (CLT)	3
Gaussian integrand $(1-x^2)^{d/2}$	Concentration of slicing measure	3.1
Normalisation by $\sqrt{\pi}$	Orthogonal compression constant	3
Orthogonal measurement bases	Slicing axis choices	4
Measurement as suppression	$R_{\text{eff}} = 1/\sqrt{d+3}$	4.1, 8
Non-commutativity $\\|[P,Q]\\| = \frac{1}{2}\|\sin 2\theta\|$	Rank-1 projection commutator	4
Born rule $p = \cos^2\theta$	Sphere geometry + Parseval + concentration	5
Complex amplitudes	Forced precession $\alpha = \pi/2 \Rightarrow J^2 = -\text{Id}$	6
No free geometric parameters	$\alpha = \pi/2$ forced; same axiom as $\sqrt{\pi}$	6.1
Phase–obstruction lockstep	Imaginary phase $\Leftrightarrow$ forced zero (parity of d)	6.5
Interference	Phase accumulation from forced precession	6.3
Unitary time evolution	Cascade propagator (discrete)	7
Schrödinger equation	Effective description from discrete propagator	7.4
Arrow of time	Asymptotic compactification per step	7, 8
Scale coincidence $R_{\text{eff}}/\sigma \to 1$	Compactification + Gaussian width	8.2
Second cascade constant $\text{erf}(1/\sqrt{2})$	Scale coincidence at $d \to \infty$	8.2
Irreversibility is asymptotic	$R_{\text{eff}} > 0$; no sharp loss	8.3
Unitarity + apparent irreversibility	Global unitary; boundary retains info	8.4
Decoherence $D = e^{-\Delta x^2/4R_{\text{eff}}^2}$	Tracing fibre with slicing weight	8.5
Cascade coherence 93.9%	Gram eigenvalue deficit $\varepsilon = 6.1\%$	8.6
$\hbar$ as quantum scale	Unit matching: $\hbar = (\pi/2)/\Delta t$	9
Classical limit	Coarse time resolution $\Rightarrow$ no phase	9.2
Tensor product structure	Iterated slicing decomposition	10
Entanglement	Non-factorisable cascade	10.2

Every row is derived from two inputs: (1) the cascade geometry of [1], and (2) the assumption that the observer has access to 4 dimensions. The compactification results (Section 8) are proved in this paper from the Beta function, building on the boundary dominance identity of [1], Theorem 3.1. The phase–obstruction lockstep (Corollary 6.6) is derived from the forced precession and sphere parity alone; the Clifford refinement to period 8 is deferred to Part III.

12 What This Paper Does and Does Not Do

Does:

- Shows that a 4D observer in the cascade geometry recovers the structural framework of quantum mechanics.
- Derives the Born rule $p = \cos^2\theta$ from the geometry of the spherical cap, the slicing integrand, and Parseval's identity on S^{d-1} (Theorem 5.2). No probability axiom enters.
- Proves the commutator of rank-1 projections: $\|[P_{e_1}, P_{e_2}]\| = \frac{1}{2}|\sin 2\theta|$ (Theorem 4.2), vanishing at $\theta \in \{0, \pi/2, \pi\}$ with maximum at $\pi/4$.
- Proves Bell inequality violation $S = 2\sqrt{2}$ on S^7 (Theorem 10.3) from the Born rule applied to tensor product directions.
- Proves that the precession angle $\alpha = \pi/2$ is forced by the cascade's orthogonality axiom, eliminating all free geometric parameters (Theorem 6.1). The proof explicitly distinguishes the two applications of the orthogonality axiom (one forcing $\sqrt{\pi}$ in the integral, one forcing α) and shows they are sequential applications of the same axiom, not a circular argument.
- Derives the Gaussian character of wavefunctions from projection, with the elementary mechanism identified: the slicing integrand $(1-x^2)^{d/2}$ is itself approximately Gaussian (Lemma 3.1).
- Provides a geometric origin for complex amplitudes via forced axis precession.
- Identifies time with the cascade's slicing direction, yielding the arrow of time and a natural lapse function.
- Derives the discrete evolution equation (Theorem 7.2) and shows the Schrödinger equation arises as the effective description for a 4D observer (Corollary 7.3), with $\mathcal{H}$ expressed as a Gamma function ratio. The approximation is valid deep in the cascade; at $d = 4$ the discrete propagator is primary.
- Proves the compactification radius $R_{\text{eff}} = 1/\sqrt{d+3}$ from the Beta function (Theorem 8.1) and develops its physical interpretation: each temporal step is an asymptotic compactification. The arrow of time, the nature of measurement, and the compatibility of unitarity with apparent irreversibility all receive concrete geometric content from this identification.
- Establishes the scale coincidence $R_{\text{eff}}/\sigma \to 1$ (Theorem 8.2) and derives the second universal cascade constant $\operatorname{erf}(1/\sqrt{2}) = 0.68269$ (Corollary 8.3).

- Shows that boundary dominance ($\Omega_{d-1}/V_d = d$) justifies the primacy of the unit sphere as the observer's state space.
- Derives the phase–obstruction lockstep (Corollary 6.6): the propagator phase is imaginary if and only if the sphere carries a hairy ball zero, both controlled by the parity of d. The cascade-native classification has period 4; the Clifford refinement to period 8 is noted and deferred to Part III.
- Derives the geometric decoherence rate (Theorem 8.7): the cascade's slicing traces out the fibre, producing Gaussian decoherence $D = \exp(-\Delta x^2/4R_{\text{eff}}^2)$ with decoherence length $2R_{\text{eff}}$. The structure matches environment-induced decoherence, with the integrated-out direction as the environment and the compactification radius as the coherence length.

Does not:

- Derive why we observe 4 dimensions. This is an empirical input.
- Derive the specific Hamiltonian of any physical system. The cascade gives the framework but not the content.
- Derive the value of $\hbar$. The geometric factor $\pi/2$ is derived; the physical time increment Δt is not.
- Derive quantum field theory. The cascade gives single-particle QM; extension to fields requires additional structure.
- Use any result from physics. The only inputs are the cascade's geometry and the observer's dimensionality.

13 Open Questions

1. **Quantum field theory.** Whether the dimensions between successive thresholds encode specific field content is open.
2. **Physical time increment.** The geometric value $\alpha = \pi/2$ is derived. The remaining question is how the physical time increment Δt is fixed, which would determine $\hbar$.
3. **Lorentzian signature.** The cascade is real and Euclidean. The forced precession $\alpha = \pi/2$ produces $J^2 = -\text{Id}$. Whether the identification $g_{tt} = (iN)^2 = -N^2$ can be derived as a theorem is open.
4. **The measurement problem.** The cascade provides a natural account of measurement as asymptotic compactification: the measured direction is suppressed to $R_{\text{eff}} = 1/\sqrt{d+3}$, not sharply projected. Whether this smooth suppression—combined with the boundary's retention of information ($\Omega/V = d$)—replaces the projection postulate is open. The structural ingredients (global unitarity, apparent local irreversibility, boundary encoding) are suggestive but not yet a derivation.

5. **Decoherence and the descent correction.** Theorems 8.7 and 8.10 derive the per-step and cumulative decoherence: $\varepsilon = 6.11\%$ over the full 213-step descent, with the cascade retaining 93.9% coherence (Corollary 8.11). This eigenvalue deficit is the same quantity that drives the descent corrections in the Part 0 Supplement. The remaining question: whether the cascade's geometric decoherence, applied to specific physical systems, reproduces the quantitative predictions of environment-induced decoherence [7].

References

[1] RTAC, *The Cascade Series — Part 0: Scale Variance from Orthogonality: How the Unit Ball Generates* 10^{120} *Orders of Magnitude*, March 2026.

[2] M. Ledoux, *The Concentration of Measure Phenomenon*, AMS Mathematical Surveys and Monographs 89, 2001.

[3] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite-Dimensional Normed Spaces*, Lecture Notes in Mathematics 1200, Springer, 1986.

[4] C. A. Fuchs, "Quantum Mechanics as Quantum Information (and only a little more)," arXiv:quant-ph/0205039, 2002.

[5] R. Clifton, J. Bub, and H. Halvorson, "Characterizing quantum theory in terms of information-theoretic constraints," *Foundations of Physics* **33**, 1561–1591 (2003).

[6] A. M. Gleason, "Measures on the closed subspaces of a Hilbert space," *Journal of Mathematics and Mechanics* **6**, 885–893 (1957).

[7] W. H. Zurek, "Quantum Darwinism," *Nature Physics* **5**, 181–188 (2009).

[8] L. Hardy, "Quantum theory from five reasonable axioms," arXiv:quant-ph/0101012, 2001.

[9] R. Arnowitt, S. Deser, and C. W. Misner, "The dynamics of general relativity," in *Gravitation: An Introduction to Current Research*, Wiley, 1962.