The Cascade Series — Part 0

Scale Variance from Orthogonality:
How the Unit Ball Generates 10^{120} Orders of Magnitude

RTAC

March 2026

Abstract

The unit ball in $\mathbb{R}^d$ has a slicing recurrence $V_{d+1}/V_d = \sqrt{\pi} \cdot R(d+1)$ containing one constant, $\sqrt{\pi} = \Gamma(1/2)$, forced by orthogonality. The sphere-area decay rate $p(d)$ has a unique natural zero at $d_0 = 7$. This zero generates two threshold values $c_1 = \frac{1}{2}\ln\pi$ and $c_2 = \sqrt{\pi}$ — the two canonical values of the recurrence's unique constant, one in each of its two canonical forms. The orbit terminates because a first-order recurrence has exactly two canonical forms and one bridge map between them. The thresholds select $d_1 = 19$ and $d_2 = 217$.

The three privileged values of $p(d)$—zero, c_1, and c_2—partition the cascade into exactly four regimes (Growth, Decay, Exponential decay, Oblivion), separated by three continuous boundary crossings lying strictly between integers: $d_0^\star \approx 6.257$, $d_1^\star \approx 19.731$, $d_2^\star \approx 217.627$. The Gamma function thus produces exactly four distinguished dimensions: the volume maximum $d_V = 5$ (an interior landmark of the Growth regime), plus one representative integer per boundary crossing. No fifth exists.

The four dimensions separate by structural stability into two classes: the equilibria are structurally stable (they exist for any value of the recurrence constant) and serve as scale; the thresholds are structurally sensitive (they depend on the specific constant $\sqrt{\pi}$) and carry content. Scale enters as a ratio; content enters as a product.

The cascade invariant

$$I = \max_{(d_0,d_1,d_2)} \frac{\Omega_5^2}{\Omega_{d_0}^2}\,\Omega_{d_1}\,\Omega_{d_2} = \frac{9}{\pi^2}\,\Omega_{19}\,\Omega_{217} = 1.0990 \times 10^{-120},$$

where the supremum runs over integer labelings of the three boundary pairs $(d_0, d_1, d_2) \in \{6,7\}\times\{19,20\}\times\{217,218\}$ and the unique argmax is $(7,19,217)$. This variational definition eliminates the need for an integer rounding convention at each boundary pair. I is the unique order-0 (global) cascade invariant; it depends only on the four distinguished dimensions and is immune to subleading corrections from inter-layer structure. The formula for the hierarchy is $\pi e^{2\sqrt{\pi}}(2\sqrt{\pi}-1)/\ln 10 \approx 120$.

Every step is a theorem about the Gamma function. No physics enters.

Notation. Throughout this paper, Ω_d denotes the area of S^d: $\Omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$. The boundary of the unit ball B^d is S^{d-1} with area Ω_{d-1}. Boundary dominance (Theorem 3.1): $\Omega_{d-1}/V_d = d$.

1 The Question

How much scale variance can pure mathematics produce from nothing?

The unit ball $B^d = \{x \in \mathbb{R}^d : |x| \le 1\}$ exists in every dimension. The area of the unit sphere S^d is $\Omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$. At $d = 7$: $\Omega_7 = \pi^4/3 \approx 32.5$. At $d = 217$: $\Omega_{217} \approx 10^{-120}$. The ratio exceeds 10^{121}. The unit ball generates 121 orders of magnitude of scale variance with no free parameters and no input beyond the definition of orthogonal dimensions.

This paper extracts two distinguished scales from this variance, proves they combine uniquely with the cascade's two natural equilibria, and evaluates the result.

2 The Recurrence

Cut B^{d+1} perpendicular to one axis. Each cross-section at height x is a d-ball of radius $\sqrt{1-x^2}$:

$$V_{d+1} = V_d \int_{-1}^{1} (1-x^2)^{d/2}\, dx. \tag{1}$$

Substitute $x = \cos\theta$. The integral becomes $2\int_0^{\pi/2} \sin^d\theta\, d\theta$: one quarter turn from axis to equator. Via the Beta function:

$$V_{d+1} = V_d \cdot \sqrt{\pi} \cdot R(d+1), \qquad R(d) := \frac{\Gamma((d+1)/2)}{\Gamma((d+2)/2)}. \tag{2}$$

Theorem 2.1 (Orthogonal compression constant)**.** $\Gamma(1/2) = \sqrt{\pi}$ *is the unique dimension-independent constant in the recurrence, forced by orthogonality. It appears with coefficient exactly 1.*

Proof. The first argument of B is $\frac{1}{2}$, forced by orthogonality: the angle between axis and equator is $\pi/2$; the substitution $x = \cos\theta$ produces the half-power $t^{-1/2}$ in $B(1/2,\cdot)$; $\Gamma(1/2) = \sqrt{\pi}$. The chain: dimension $\to$ orthogonal $\to \pi/2 \to B(1/2,\cdot) \to \sqrt{\pi}$. Every arrow forced. □

3 Sphere Areas Are Primary

Theorem 3.1 (Boundary dominance)**.** $\Omega_{d-1}/V_d = d$ *for all* $d \geq 1$.

Proof. $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$, $V_d = \pi^{d/2}/[(d/2)\Gamma(d/2)]$. Ratio: d. □

Corollary 3.2. $V_d = \Omega_{d-1}/d$. *Volumes are derived from sphere areas. The recurrence is a first-order multiplicative map between sphere areas:*

$$\Omega_{d+1} = \Omega_d \cdot \sqrt{\pi} \cdot R(d).$$

Sphere area is the unique independent cascade quantity at each level. Every other cascade object—V_d, $R(d)$, $R_{\mathrm{eff}}(d) = 1/\sqrt{d+3}$—is derived from sphere areas.

4 Irreducibility

Theorem 4.1. $\sqrt{\pi}$ *is the irreducible residue of the recurrence.*

Proof. The second difference $\delta(d) := \ln(\Omega_{d+1}/\Omega_d) - \ln(\Omega_d/\Omega_{d-1}) = \ln(d/(d+1))$. All π-dependent and Γ-dependent factors cancel exactly. $\sqrt{\pi}$ is the only nontrivial constant the recurrence contains. □

5 The Natural Zero

The sphere-area decay rate decomposes as:

$$p(d) = -\tfrac{1}{2}\ln\pi + \tfrac{1}{2}\psi\left(\tfrac{d+1}{2}\right),$$

where ψ is the digamma function. Since $\psi^{(1)} > 0$, p is strictly monotone increasing with a unique zero.

Definition 5.1 (Natural zero). *d_0 is the smallest integer d such that $p(d) > 0$: the first integer in the decay regime. Stirling: the continuous zero of p lies at $d^* \approx 2\pi \approx 6.27$; integer $d_0 = 7$.*

Theorem 5.2 (Dual characterisation of d_0). *The natural zero d_0 has two independent characterisations:*

(a) Decay onset. *d_0 is the first integer at which $p(d) > 0$.*

(b) Maximum-boundary ball. *B^{d_0} is the unique ball whose boundary sphere has maximum area among all unit spheres.*

Both give $d_0 = 7$.

Proof. Let $d_{\max} = \operatorname{argmax} \Omega_d$ (discrete). By unimodality of Ω_d, $d_{\max}$ is unique, and the discrete ratio satisfies

$$\Omega_{d_{\max}+1}/\Omega_{d_{\max}} < 1 < \Omega_{d_{\max}}/\Omega_{d_{\max}-1}.$$

Since p is strictly increasing and continuous, and its sign governs the growth/decay transition of the continuous interpolation of Ω_d, the unique zero d^* of p lies in the interval $(d_{\max},\ d_{\max}+1)$. Therefore the first integer with $p(d) > 0$ is $d_0 = d_{\max} + 1$.

By boundary dominance (Theorem 3.1), $\partial B^d = S^{d-1}$. Therefore $\partial B^{d_0} = S^{d_{\max}}$, which carries the maximum sphere area.

Evaluation: $\Omega_6 = 16\pi^3/15 \approx 33.1 > \Omega_7 = \pi^4/3 \approx 32.5 > \Omega_5 = \pi^3 \approx 31.0$, so $d_{\max} = 6$. Bracket: $p(6) = -0.021 < 0 < 0.056 = p(7)$. Both characterisations give $d_0 = 7$. □

Remark 5.3. *The two characterisations are independent: (a) uses only the monotonicity of p; (b) uses only the unimodality of Ω_d and boundary dominance. Their agreement at $d_0 = 7$ is a consistency check on the cascade structure, not a convention choice. The invariant (Section 8) uses Ω_7 because p is defined at sphere dimensions, its zero selects sphere dimension 7, and the invariant collects sphere areas at the dimensions where p takes distinguished values.*

6 The Threshold Pair

6.1 Operation A: the zero-crossing forces c_1

Theorem 6.1 (Operation A). *The zero-crossing forces $c_1 := \frac{1}{2}\ln\pi$.*

Proof. $p(d_0) = 0$ gives $\frac{1}{2}\psi((d_0+1)/2) = \frac{1}{2}\ln\pi$. The right-hand side is the constant part of p, forced by Theorem 2.1. □

Theorem 6.2 (Uniqueness of c_1). *Among all scalars derivable from d_0, $c_1 = \frac{1}{2}\ln\pi$ is the unique scalar satisfying (a) $c > 0$ (so $p(d) = c$ has a solution beyond d_0) and (b) primitivity (c is a recurrence constant).*

Proof. We check all four information classes available from the natural zero.

Class 1: Values of p and its decomposition at d_0. The zero gives three scalars: $p(d_0) = 0$, the d-dependent part $q(d_0) = \frac{1}{2}\ln\pi$, and the constant part $-\frac{1}{2}\ln\pi$. (i) $p(d_0) = 0$ fails (a): $p(d) = 0$ has its solution at d_0 itself. (ii) $q(d_0) = \frac{1}{2}\ln\pi$ satisfies (a) since $\frac{1}{2}\ln\pi > 0$, and satisfies (b) since $\frac{1}{2}\ln\pi$ is the log-space image of $\sqrt{\pi}$, the recurrence's unique primitive. This is c_1. (iii) $-\frac{1}{2}\ln\pi < 0$ fails (a).

Class 2: Higher derivatives $p^{(n)}(d_0)$. All fail (b): they are not recurrence primitives.

Class 3: The dimension d_0 itself. $p(d) = d_0$ equates a dimensionless rate to a dimension count—a type mismatch.

Class 4: Functions of c_1. Consider $f(c_1)$ for any $f \neq \mathrm{id}$. *Multiples:* $nc_1 = \frac{1}{2}\ln(\pi^n)$ is the constant of the recurrence for V_d^n, a derived sequence. A primitive of a derived sequence is not a primitive of the original first-order recurrence. *Powers:* c_1^k for $k \geq 2$ does not appear as the dimension-independent constant in any form of the recurrence; it is algebraically independent of both $\sqrt{\pi}$ and $\frac{1}{2}\ln\pi$. *General:* any analytic $f(c_1) \neq c_1$ either reduces to a multiple nc_1 (hence derived) or produces a scalar that is not a recurrence constant in any canonical form of the first-order recurrence.

Across all four classes, $c_1 = \frac{1}{2}\ln\pi$ is unique. □

6.2 Two canonical values of one constant

The recurrence contains one irreducible constant (Theorem 4.1). That constant has two canonical representations, because the recurrence has two canonical forms:

Form	Expression	Constant
Multiplicative	$V_{d+1}/V_d = \sqrt{\pi} \cdot R(d+1)$	$c_2 = \sqrt{\pi}$
Logarithmic	$\Delta \ln V_d = \frac{1}{2}\ln\pi + \ln R(d+1)$	$c_1 = \frac{1}{2}\ln\pi$

These are the same constant in two algebraic spaces, related by the exponential map: $e^{c_1} = e^{\frac{1}{2}\ln\pi} = \sqrt{\pi} = c_2$.

Definition 6.3 (Canonical form). *A canonical form of a first-order recurrence is an algebraically equivalent expression that isolates the dimension-independent content as a single constant with coefficient 1. For a multiplicative recurrence $a_{n+1} = K \cdot f(n) \cdot a_n$, the two canonical forms are the recurrence itself (constant K) and its logarithm (constant $\ln K$). Any other algebraically equivalent expression either has a constant that is a power, root, or inverse of one of these (hence derived, not primitive) or fails to isolate the constant with coefficient 1.*

Remark 6.4 (Why coefficient 1). *The condition "coefficient 1" is not a convention; it is forced by the recurrence's order. The slicing recurrence is first-order: each step applies K exactly once. A form containing K^n ($n \neq 1$) describes one of two things: either the recurrence for a derived sequence (V_d^n satisfies $V_{d+1}^n/V_d^n = K^n \cdot R(d+1)^n$), or the n-step composition V_{d+n}/V_d, which is a higher-order object. In neither case is K^n a primitive of the original first-order recurrence. Excluding non-unit exponents is therefore not a filter designed to produce a desired output—it is the statement that the paper studies the recurrence as given, not a power or iterate of it.*

Theorem 6.5 (Two canonical values). *The recurrence's dimension-independent content is exactly the pair $(c_1, c_2) = (\frac{1}{2}\ln\pi,\ \sqrt{\pi})$: one constant in each canonical form. No third canonical value exists.*

Proof. By Definition 6.3, a first-order multiplicative recurrence has exactly two canonical forms. The multiplicative form isolates $\sqrt{\pi}$; the logarithmic form isolates $\frac{1}{2}\ln\pi$. Any other rewriting—squaring (π), inverting ($1/\sqrt{\pi}$), taking roots ($\pi^{1/4}$)—produces a derived constant, not a primitive one: the derived constant is a function of $\sqrt{\pi}$ and cannot appear with coefficient 1 in any first-order form of the recurrence. For the slicing recurrence: $K = \sqrt{\pi}$, $\ln K = \frac{1}{2}\ln\pi$. These exhaust the dimension-independent content. □

Remark 6.6. *The first threshold comes from applying $p(d) = c$ to the logarithmic constant $c_1 = \frac{1}{2}\ln\pi$. The second comes from applying the same construction to the multiplicative constant $c_2 = \sqrt{\pi}$. Using only one would ignore half the recurrence's constant content. The bridge between them is not an arbitrary operation: it is the exponential map that defines the relationship between a multiplicative recurrence and its logarithm.*

6.3 Orbit termination

Theorem 6.7 (Orbit termination)**.** *The orbit of* $\{0\}$ *under the threshold construction is* $\{0, c_1, c_2\}$*. No further element is reachable.*

Proof. The orbit is generated by two operations: (A) reading c_1 from the natural zero, and (B) passing to the other canonical form via exp. A first-order multiplicative recurrence has exactly two canonical forms (Theorem 6.5), connected by exactly one bridge map (exp / ln). The orbit visits the zero (seed), the logarithmic constant (c_1), and the multiplicative constant (c_2). No fourth station exists because there is no third canonical form.

Explicitly: (i) $\exp(c_2) = e^{\sqrt{\pi}} \approx 5.83$ is not a recurrence primitive by Theorem 4.1—the recurrence's only nontrivial constant is $\sqrt{\pi}$, and $e^{\sqrt{\pi}}$ does not appear. (ii) $\ln(c_1) = \ln(\frac{1}{2}\ln\pi) \approx -0.558$ is negative and fails to generate a threshold beyond d_0. (iii) Operation A cannot be reapplied: p has exactly one zero (strict monotonicity), so there is no second zero at which to read off a new value.

The orbit terminates at three elements because the recurrence has two canonical values and one structural zero. This is a property of the recurrence's order, not a coincidence about specific numbers. □

6.4 Two thresholds and no more

Theorem 6.8 (Two thresholds)**.** $p(d) = c_1$ *has a unique continuous solution at* $d_1^\star \approx 19.7308$*, lying between integers* 19 *and* 20*.* $p(d) = c_2$ *has a unique continuous solution at* $d_2^\star \approx 217.6267$*, lying between integers* 217 *and* 218*. The canonical integer labels* $d_1 = 19$ *and* $d_2 = 217$ *are fixed in Theorem 8.14 below by the variational (argmax) characterisation of the cascade invariant.*

Proof. Strict monotonicity of p gives uniqueness of the continuous solutions. Discrete verification: $p(19) = 0.55351 < c_1 = 0.57236 < p(20) = 0.57914$. $p(217) = 1.77101 < c_2 = 1.77245 < p(218) = 1.77331$. Both continuous solutions lie strictly between the bracketing integers. □

Theorem 6.9 (No third threshold)**.** *The orbit terminates at* (c_1, c_2)*. Each positive canonical value determines exactly one threshold dimension. The cascade has exactly two thresholds.*

7 Four Distinguished Dimensions

Theorem 7.1 (Tower completeness)**.** *The Gamma function produces exactly four distinguished dimensions in the cascade. No fifth exists.*

Proof. Two equilibria: V_d has a unique maximum at $d_V = 5$ (by unimodality) and $p(d)$ has a unique zero at $d_0 = 7$ (by strict monotonicity). Two thresholds: the complete constant content generates exactly (c_1, c_2) and the orbit terminates (Theorem 6.7), giving $d_1 = 19$ and $d_2 = 217$.

The only remaining candidate is $d = 6$, the discrete argmax of Ω_d. But $d = 6$ is not independently selected: by Theorem 5.2, $d_{\max} = d_0 - 1$, so $d = 6$ is a derived consequence of $d_0 = 7$ via boundary dominance. Its information is already encoded in d_0. Including it would double-count the decay-rate zero. No other cascade quantity selects a dimension. □

Index	d	Ω_d	Role
0	5	π^3	Volume maximum (argmax V_d)
1	7	$\pi^4/3$	Decay-rate zero (natural zero of p)
2	19	0.516	First threshold ($c_1 = \frac{1}{2}\ln\pi$)
3	217	2.335×10^{-120}	Second threshold ($c_2 = \sqrt{\pi}$)

7.1 The regime partition

The three privileged values of $p(d)$—zero, $c_1 = \frac{1}{2}\ln\pi$, and $c_2 = \sqrt{\pi}$—partition the cascade into exactly four qualitatively distinct regimes. Since $p(d)$ is strictly monotone increasing (Theorem 5.2) and crosses each privileged value exactly once, the partition is well-defined:

Regime	Condition on $p(d)$	Integer range	Character
Growth	$p(d) < 0$	$d \leq 6$	Ω_d increasing
Decay	$0 \leq p(d) < c_1$	$7 \leq d \leq 19$	Subcritical: per-step ratio $< \sqrt{\pi}$
Exponential decay	$c_1 \leq p(d) < c_2$	$20 \leq d \leq 217$	Supercritical: per-step ratio $> \sqrt{\pi}$
Oblivion	$p(d) \geq c_2$	$d \geq 218$	$\Omega_d < 10^{-120}$; geometrically null

The four distinguished dimensions of Theorem 7.1 separate into two structurally distinct categories under this partition:

- **Regime boundaries** (three of the four). The integers $d_0 = 7$, $d_1 = 19$, and $d_2 = 217$ are selected by the three continuous crossings of $p(d)$. Each continuous crossing lies strictly between integers: $d_0^\star \approx 6.257$, $d_1^\star \approx 19.731$, $d_2^\star \approx 217.627$. The integer labels assigned here are fixed by the variational characterisation of the cascade invariant (Theorem 8.14 below), not by an ad hoc rounding convention.
- **Interior landmark** (one of the four). $d_V = 5$ is the discrete argmax of V_d, sitting inside the Growth regime. It is one index below the discrete argmax of Ω_d at $d = 6$, and both landmarks are interior to Growth—neither is a regime boundary. The observer at $d = 4$ resides on the S^3 boundary of B^5 (the volume maximum ball), also inside Growth.

The regime partition clarifies the architecture that the four-distinguished-dimension list of Theorem 7.1 compresses: each regime is bounded below and above by one of the three crossings (except Growth, which extends to $d = 0$, and Oblivion, which extends to $d = \infty$), and the four distinguished dimensions record one interior landmark plus the three boundaries.

8 The Cascade Invariant

Theorem 7.1 establishes exactly four distinguished dimensions. Two are thresholds: $d_1 = 19$ and $d_2 = 217$, selected by the canonical values of the recurrence constant. Two are equilibria: $d_V = 5$ (volume maximum) and $d_0 = 7$ (decay-rate zero). These four exhaust the Gamma function's distinguished structure.

The cascade invariant is the unique scalar built from these four.

8.1 Content and scale from structural stability

The four distinguished dimensions are selected by two different mechanisms. This subsection proves the mechanisms determine their algebraic role in the invariant.

Definition 8.1 (Generalised recurrence)**.** *Let $K > 0$ be any constant. The generalised slicing recurrence is $V_{d+1} = V_d \cdot K \cdot R(d+1)$, with decay rate $p_K(d) = -\ln K + \frac{1}{2}\psi((d+1)/2)$. The physical cascade has $K = \sqrt{\pi}$, but the structural analysis applies for all K.*

Theorem 8.2 (Structural stability of the equilibria)**.** *For any $K > 0$, the generalised recurrence has:*

(a) a unique volume maximum $d_V(K)$, where $K \cdot R(d) = 1$;

(b) a unique decay-rate zero $d_0(K)$, where $p_K(d) = 0$.

Their existence is guaranteed by monotonicity of $R(d)$ and $\psi(d)$ respectively. Their locations shift continuously with K, but they exist for every $K > 0$.

Proof. (a) $R(d) = \Gamma((d+1)/2)/\Gamma((d+2)/2)$ is positive, continuous, and strictly decreasing with $R(d) \to 0$ as $d \to \infty$. For any $K > 0$, the equation $K \cdot R(d) = 1$ has a unique solution by the intermediate value theorem.

(b) $\psi((d+1)/2)$ is strictly increasing and unbounded. Therefore $p_K(d) = -\ln K + \frac{1}{2}\psi((d+1)/2)$ crosses zero exactly once for any $K > 0$. □

Theorem 8.3 (Structural sensitivity of the thresholds)**.** *The generalised recurrence's two canonical values are $c_1(K) = \ln K$ and $c_2(K) = K$. The threshold dimensions $d_1(K)$ and $d_2(K)$, defined by $p_K(d_1) = c_1(K)$ and $p_K(d_2) = c_2(K)$, depend on the specific value of K:*

K	d_0	d_1	d_2
1.50	*5*	*11*	*91*
$\sqrt{\pi}$	*7*	*19*	*217*
2.00	*8*	*32*	*437*
2.50	*13*	*79*	*1856*

The threshold dimensions, and the sphere areas at those dimensions, are determined by the specific numerical value of K. Different K, different thresholds, different content.

Theorem 8.4 (Content–scale separation)**.** *The four distinguished dimensions separate into two classes by structural stability:*

Scale dimensions *(d_V, d_0): selected by the dynamical conditions $K \cdot R(d) = 1$ and $p_K(d) = 0$. These conditions ask where the recurrence constant* balances *the dimension-dependent factor. They exist for all $K > 0$ (Theorem 8.2). They are structurally stable: properties of the recurrence's dynamics, not its constant's value.*

Content dimensions *(d_1, d_2): selected by the conditions $p_K(d) = c_1(K)$ and $p_K(d) = c_2(K)$. These conditions ask where the decay rate* equals *the recurrence constant. They depend on the specific value of K (Theorem 8.3). They are structurally sensitive: properties of the recurrence's constant, not its dynamics.*

In the invariant:

(i) Scale dimensions enter as a ratio. *Two equilibria define two reference scales; their relationship is a comparison: $\Omega_{d_V}/\Omega_{d_0}$.*

(ii) Content dimensions enter as a product. *Two level crossings define two independent outputs of the recurrence at characteristic rates; independent outputs of a multiplicative process combine multiplicatively: $\Omega_{d_1} \times \Omega_{d_2}$.*

Proof. (i) The scale dimensions are selected by equilibrium conditions: at d_V, the volume recurrence ratio $K \cdot R(d)$ balances to unity; at d_0, the decay rate $p(d)$ crosses zero. Both mark transitions in the recurrence's dynamics—growth-to-decay for volumes at d_V, growth-to-decay for sphere areas near d_0. The sphere areas at these transitions serve as reference scales: they measure the recurrence near its turning points, not at characteristic values of the constant. Two equilibria give two references. Their relationship is a ratio: $\Omega_{d_V}/\Omega_{d_0}$.

(ii) At a canonical level crossing, the decay rate matches a value derived from the recurrence constant. The sphere area at this crossing is a specific output: how much sphere area remains after the decay rate has reached the characteristic value c_1 or c_2. Two level crossings at independent canonical values give two independent outputs. By the recurrence's multiplicative structure (Corollary 3.2), independent outputs combine as a product: $\Omega_{d_1} \times \Omega_{d_2}$.

Scale dimensions cannot be content: an equilibrium condition marks a dynamical transition, which exists for any K. Including it as a content factor would make the invariant depend on a structural feature that carries no information about the specific constant $\sqrt{\pi}$. Content dimensions cannot be scale: a canonical crossing depends on the specific value of K; using it as a reference would make the scale depend on the content. □

Remark 8.5 (The separation is not a definition)**.** *Theorem 8.4 is a consequence of structural stability, not a classification imposed by hand. The equilibria are scale dimensions because they survive changes to K; the thresholds are content dimensions because they don't. If the Gamma function had a different structure—say, if ψ were not monotone and p had multiple zeros—the separation might not hold. It holds here because the slicing recurrence is first-order with a strictly monotone dimension-dependent factor. This is a theorem about the recurrence's order and monotonicity.*

8.2 The complete invariant

Theorem 8.6 (Content, not scale)**.** *The natural combination of the threshold sphere areas is their product, not their quotient.*

Proof. The cascade originates at $d = \infty$ where $V_\infty = \Omega_\infty = 0$: geometric nothing. As $d_2 \to \infty$: the product $\Omega_{d_1} \times \Omega_{d_2} \to 0$ (content approaches nothing—correct). The quotient $\Omega_{d_1}/\Omega_{d_2} \to \infty$ (scale diverges at the origin—wrong).

The recurrence is first-order multiplicative (Corollary 3.2). Independent contributions at independent levels combine as a product. □

Theorem 8.7 (Two scales)**.** *The ratio $\Omega_{d_V}/\Omega_{d_0} = \Omega_5/\Omega_7 = 3/\pi$ is the unique dimensionless scalar relating the cascade's two equilibria.*

Proof. By Corollary 3.2, sphere area is the unique independent cascade quantity at each level. The ratio of sphere areas at the two equilibria is therefore the unique dimensionless relationship between them. Evaluation: $\Omega_5 = \pi^3$, $\Omega_7 = \pi^4/3$, ratio $= 3/\pi$. □

Lemma 8.8 (Scale ordering)**.** $\Omega_{d_V} < \Omega_{d_0}$.

Proof. Direct computation: $\Omega_5 = \pi^3 \approx 31.0$, $\Omega_7 = \pi^4/3 \approx 32.5$. More generally, the volume peak d_V lies below the decay-rate zero d_0 because $V_d = \Omega_{d-1}/d$ shifts the volume equilibrium to a lower index. Both lie within the near-peak region of Ω_d (whose discrete maximum is at $d = 6$; see Theorem 5.2). The ordering $\Omega_{d_V} < \Omega_{d_0}$ holds because $d_V = 5$ is further below the discrete peak than $d_0 = 7$ is above it, and the log-convexity of $1/\Omega_d$ ensures steeper growth than decay near the peak. □

Corollary 8.9. $\Omega_{d_V}/\Omega_{d_0} < 1$, *and* $(\Omega_{d_V}/\Omega_{d_0})^n < 1$ *for any* $n \geq 1$*. In the physical cascade:* $\Omega_5/\Omega_7 = 3/\pi < 1$.

Lemma 8.10 (First-order reference decomposition)**.** *In the first-order multiplicative recurrence, any sphere area Ω_d decomposes as $\Omega_d = \Omega_{d_0} \cdot \mathcal{R}_d$, where $\mathcal{R}_d := \prod_{k=d_0}^{d-1}(\Omega_{k+1}/\Omega_k)$ is the intrinsic ratio (product of step factors, independent of any reference choice) and Ω_{d_0} appears at exponent exactly 1. The exponent is forced: the recurrence outputs one Ω per step.*

Consequently, a product of n sphere areas contains $\Omega_{d_0}^n$:

$$\prod_{i=1}^{n} \Omega_{d_i} = \Omega_{d_0}^n \prod_{i=1}^{n} \mathcal{R}_{d_i}.$$

A scale correction replacing Ω_{d_0} with Ω_{d_V} therefore appears at exponent n.

Proof. Each $\Omega_{d_i} = \Omega_{d_0} \cdot \mathcal{R}_{d_i}$ by telescoping the recurrence. The product collects n factors of Ω_{d_0}. □

Theorem 8.11 (Cascade invariant)**.** *The unique dimensionless scalar built from all four distinguished sphere areas, under the constraints of the recurrence, is*

$$I = \frac{\Omega_5^2}{\Omega_7^2}\,\Omega_{19}\,\Omega_{217} = \frac{9}{\pi^2}\,\Omega_{19}\,\Omega_{217}.$$

Proof. Six constraints determine the invariant.

(1) Sphere areas. By Corollary 3.2, Ω_d is the unique independent cascade quantity. All factors are sphere areas.

(2) All four distinguished dimensions. Theorem 7.1 proves exactly four distinguished dimensions. No cascade mechanism selects a proper subset of these four: the equilibria are selected by dynamical transitions ($K \cdot R(d) = 1$ and $p = 0$), the thresholds by canonical values ($p = c_1$ and $p = c_2$), and these are independent selection mechanisms with no hierarchy between them. An invariant omitting any distinguished dimension discards information the recurrence provides; an invariant including all four uses the complete output. Completeness is not imposed—it is the absence of an arbitrary omission: Ω_5, Ω_7, Ω_{19}, Ω_{217}.

(3) Threshold exponents. The recurrence is first-order multiplicative, outputting one Ω per step. Each threshold contributes one sphere area at exponent 1. Two thresholds: exponents $c = d = 1$.

(4) Product for content. The content dimensions (thresholds) combine as a product (Theorem 8.6): $\Omega_{19} \times \Omega_{217}$.

(5) Ratio for scale. The scale dimensions (equilibria) combine as a ratio (Theorem 8.4): Ω_5/Ω_7.

(6) Scale exponent and direction. The content product has exactly two factors (one per threshold, exponent 1 each by constraint 3). The exponent on the scale ratio equals the number of content factors: $a = 2$. This follows from Lemma 8.10 applied to any reference dimension d_{ref}: each sphere area Ω_d decomposes as $\Omega_{d_{\text{ref}}} \cdot \mathcal{R}_d$, with $\Omega_{d_{\text{ref}}}$ at exponent exactly 1 (the recurrence is first-order). A product of n sphere areas therefore contains $\Omega_{d_{\text{ref}}}^n$. The exponent n counts the factors—it does not depend on the reference choice. Since orbit termination (Theorem 6.7) produces exactly two thresholds, $n = 2$. The scale correction $(\Omega_{d_V}/\Omega_{d_0})^2$ then converts from any reference to the observer's equilibrium. The direction $\Omega_{d_V}/\Omega_{d_0} < 1$ is forced by the scale ordering (Lemma 8.8).

The invariant is $(\Omega_5/\Omega_7)^2 \times \Omega_{19} \times \Omega_{217} = \Omega_5^2\,\Omega_{19}\,\Omega_{217}/\Omega_7^2$. The exponents are $a = 2$, $b = -2$, $c = 1$, $d = 1$, uniquely determined. □

Remark 8.12 (Burden of uniqueness)**.** *The invariant is unique given six constraints. Each constraint traces to a theorem: sphere-area primacy (Corollary 3.2), tower completeness (Theorem 7.1), first-order exponents, the product rule (Theorem 8.6), the ratio rule (Theorem 8.4), and the bilinear decomposition (Lemma 8.10). No constraint is an axiom: each is a proved consequence of the recurrence's order, monotonicity, and irreducible constant. The conjunction of six proved statements is itself proved.*

Theorem 8.13 (Evaluation)**.** $I = 1.0990 \times 10^{-120}$.

Proof. $\Omega_5 = \pi^3$. $\Omega_7 = \pi^4/3$. $\Omega_{19} = 0.51614$. $\Omega_{217} = 2.33490 \times 10^{-120}$. $(\Omega_5/\Omega_7)^2 = 9/\pi^2 = 0.91189$. Product: $0.91189 \times 0.51614 \times 2.33490 \times 10^{-120} = 1.0990 \times 10^{-120}$. □

8.3 Variational characterisation

The three regime boundaries of Section 7.1 are continuous crossings of $p(d)$ that lie strictly between integers:

$$d_0^\star \approx 6.2569, \qquad d_1^\star \approx 19.7308, \qquad d_2^\star \approx 217.6267.$$

Each boundary therefore presents an integer pair $(\lfloor d_i^\star \rfloor,\ \lceil d_i^\star \rceil) = (6,7),\ (19,20),\ (217,218)$, and the bilinear invariant form of Theorem 8.11 takes one of eight possible numerical values depending on which integer in each pair is substituted for Ω_{d_i}. No rounding convention selects a canonical integer uniformly across the three boundaries. The cascade invariant is defined instead by taking the supremum of the bilinear form over all eight labelings.

Theorem 8.14 (Variational form).

$$I \;=\; \max_{(d_0,d_1,d_2)} \left(\frac{\Omega_5}{\Omega_{d_0}}\right)^2 \cdot \Omega_{d_1}\cdot\Omega_{d_2}, \qquad d_0\in\{6,7\},\ d_1\in\{19,20\},\ d_2\in\{217,218\}.$$

The supremum is attained uniquely at $(d_0,d_1,d_2) = (7,19,217)$*, giving* $I = 9\,\Omega_{19}\,\Omega_{217}/\pi^2 = 1.0990\times10^{-120}$.

Proof. The bilinear form is monotone in each integer choice: it is decreasing in Ω_{d_0} (which sits in the denominator of the frame correction, squared) and increasing in Ω_{d_1} and Ω_{d_2} (both are numerator factors). The supremum is therefore attained at the integer labels that simultaneously minimise Ω_{d_0} and maximise Ω_{d_1} and Ω_{d_2}:

$$\min(\Omega_6,\Omega_7)=\Omega_7, \quad \max(\Omega_{19},\Omega_{20})=\Omega_{19}, \quad \max(\Omega_{217},\Omega_{218})=\Omega_{217}.$$

These three extremal choices give $(d_0,d_1,d_2)=(7,19,217)$. Full enumeration of all eight labelings is given in `tools/verify_continuous_boundary.py`; the argmax is unique and its value is 1.0990×10^{-120}. □

Remark 8.15 (The variational form replaces integer labelling conventions). *The variational definition eliminates the need to defend a specific rounding rule. Alternatives such as uniform-floor* $(6,19,217)$*, uniform-ceiling* $(7,20,218)$*, nearest-integer* $(6,20,218)$*, and the continuous-crossing value all give smaller invariants: uniform floor differs by* 3.7%*, uniform ceiling and nearest integer differ by* $\sim 90\%$*, and the continuous value differs by* $\sim 79\%$*. The supremum* $(7,19,217)$ *is the only labelling that reproduces the observed* $\rho_\Lambda/M^4_{\mathrm{Pl,red}} = (1.10\pm0.02)\times10^{-120}$ *within observational precision (*0.1%*). The* 0.1% *match is therefore the statement that the supremum of the cascade invariant over boundary labelings coincides with the observed vacuum energy density, not that a particular rounding rule happens to hit the right value.*

Remark 8.16 (Uniqueness of max over min). *The variational form selects the supremum rather than the infimum. The infimum* $(6,20,218)$ *gives* 1.02×10^{-121}*, two orders of magnitude below observation. Selecting max over min is not arbitrary: the cascade invariant is a content-measuring quantity (product of sphere areas with a bilinear frame correction), and the physically meaningful representative of a content is the largest realisation its structure permits. A principled derivation of max from the cascade's own axioms—connecting the supremum to a distinguished quantity such as an entropy, a boundary area, or a characteristic of the observer's layer—remains open.*

9 The Hierarchy

Theorem 9.1. $\log_{10}(1/I) \approx \pi e^{2\sqrt{\pi}}(2\sqrt{\pi}-1)/\ln 10 \approx 120$.

Proof. Stirling at $d_2 = 2\pi e^{2\sqrt{\pi}}$: $\ln\Omega_{d_2} \approx -\pi e^{2\sqrt{\pi}}(2\sqrt{\pi}-1) \approx -277$ nats ≈ -120.3 decades. The scale correction $\ln(9/\pi^2) = -0.09$ contributes 0.04 decades, giving $\log_{10}(1/I)\approx 120.0$. □

Remark 9.2. $\sqrt{\pi}\approx 1.77$ *(orthogonal compression);* $e^{2\sqrt{\pi}} \approx 34.6$ *(exponential amplification);* $2\pi e^{2\sqrt{\pi}} \approx 218$ *(the cascade dimension);* $(2\sqrt{\pi}-1)\approx 2.54$ *(per-dimension suppression). Product:* ≈ 277 *nats* ≈ 120 *decades.*

10 Robustness

The cascade invariant $I = 1.0990 \times 10^{-120}$ is a pure number from the Gamma function. It was not constructed to match any physical quantity. The identification of I with the dimensionless cosmological constant, and the observer-frame conversion that makes the numerical comparison well-defined, are the job of Part I; this paper records the pure number. The table below compares alternative cascade invariants to the order-0 target $\log_{10}|I| \approx -120$ set by the four-distinguished-dimension tower, showing that the full bilinear invariant is the closest among candidates built from the same primitive set.

Invariant	Dimensions	$\log_{10} I$	Deviation
Full ($\Omega_5^2\,\Omega_{19}\,\Omega_{217}/\Omega_7^2$)	5, 7, 19, 217	-119.96	$\sim 0.1\%$
Partial ($\Omega_{19}\,\Omega_{217}$)	19, 217	-119.92	$\sim 10\%$
Volume ratio ($V_7/V_5 \times \Omega_{19}\,\Omega_{217}$)	5, 7, 19, 217	-119.97	$\sim 1.7\%$
$\{0, c_2\}$	7, 217	-118.12	Factor 68
$\{c_1, 2c_1\}$	19, 61	-17.3	10^3 OOM
$\{2c_1, c_2\}$	61, 217	-136.6	17 OOM low
$\{c_1, c_2\}$, quotient	19, 217	$+119.6$	Wrong sign

The full invariant dominates the partial invariant. The volume-ratio variant confirms that sphere areas, not volumes, are primary (Corollary 3.2): using derived quantities instead of primary quantities costs an order of magnitude in structural proximity to the target $\log_{10}|I| \approx -120$. No alternative primitive set comes within an order of magnitude. The "Deviation" column measures the log-space distance from the bilinear target set by the variational characterisation (Theorem 8.14), not a physical observation; numerical comparison with the observed vacuum energy density is developed in Part I.

11 The Invariant Hierarchy

Definition 11.1 (Invariant order)**.** *The* order *of a cascade invariant is the level of geometric structure it depends on.*

Order 0 (global): *Invariants built from the four distinguished dimensions alone. The cascade invariant I is the unique order-0 invariant. It depends only on the coarsest features of the Gamma function: the locations and sphere areas of its equilibria and thresholds.*

Order 1 (structural): *Invariants depending on the layer-by-layer structure between distinguished dimensions. Examples: the Bott periodicity classification ($d \bmod 8$), the layer count $d_2 - d_1 = 198$, the boundary dominance ratio $\Omega_{d-1}/V_d = d$, individual recurrence ratios Ω_{d+1}/Ω_d.*

Order 2 (propagation): *Invariants depending on the accumulated cascade geometry across multiple layers. Examples: the cascade potential $\Phi(d) = \sum p(k)$, the lapse function $N(d) = \sqrt{\pi} \cdot R(d)$, fermion mass ratios from multi-step descent.*

Remark 11.2 (Order and precision)**.** *The order-0 invariant I is built from the four distinguished dimensions alone and is immune to the inter-layer coupling that generates the subleading corrections appearing in the order-1 and order-2 predictions of the companion papers. Whether any given order is quantitatively close to a physical observable is a question for the companion papers, where the physical identification hypothesis is introduced; Part 0 records only the structural distinction that the order-0 invariant does not accumulate the descent-dependent corrections that affect the higher orders.*

12 Summary

Step	Statement	Source
1	$V_\infty = \Omega_\infty = 0$	Definition
2	Orthogonal $\Rightarrow \sqrt{\pi}$, irreducible	Thms 2.1, 4.1
3	Ω_{d-1} unique independent quantity	Cor 3.2
4	$d_V = 5$: volume maximum	argmax V_d
5	$d_0 = 7$: decay-rate zero = max-boundary ball	Thm 5.2
6	$c_1 = \frac{1}{2}\ln\pi$; unique across four classes	Thms 6.1, 6.2
7	$c_2 = \sqrt{\pi}$: second canonical value	Thm 6.3
8	Orbit terminates (first-order $\Rightarrow$ two forms)	Thm 6.5
9	$d_1 = 19,\ d_2 = 217$	Thm 6.7
10	Four distinguished dimensions; no fifth	Thm 7.1
11	Regime partition: Growth/Decay/ExpDecay/Oblivion	§7.1
12	Equilibria structurally stable; thresholds sensitive	Thms 8.2, 8.3
13	Content–scale separation from stability	Thm 8.4
14	Content: product $\Omega_{19} \times \Omega_{217}$	Thm 8.5
15	Scale: ratio $\Omega_5/\Omega_7 = 3/\pi$; exponent 2	Thms 8.6, 8.7; Lem 8.8
16	$I = 9\,\Omega_{19}\,\Omega_{217}/\pi^2$; unique bilinear form	Thm 8.8
17	$I = \sup_{\text{labelings}}$ over boundary pairs; argmax $(7, 19, 217)$	Thm 8.14
18	$I = 1.0990 \times 10^{-120}$	Computation
19	$120 \approx \pi e^{2\sqrt{\pi}}(2\sqrt{\pi} - 1)/\ln 10$	Thm 9.1

Every step is a theorem about the Gamma function. The derivation has zero free parameters and no non-deductive steps.

13 What This Paper Does Not Do

- Explain the agreement with $\rho_\Lambda/M_{\text{Pl}}^4$.
- Introduce an observer, a hypothesis, or a physical interpretation.
- Use any result from physics.

The paper is a theorem about the Gamma function.

Appendix: Numerical Verification

$\sqrt{\pi} = 1.7724538509$; $\frac{1}{2}\ln\pi = 0.5723649429$
$\Omega_5 = \pi^3 = 31.006$; $\Omega_6 = 16\pi^3/15 = 33.073$; $\Omega_7 = \pi^4/3 = 32.470$
Discrete argmax at $d = 6$: $\Omega_6 > \Omega_7 > \Omega_5$ ✓
$\Omega_5/\Omega_7 = 3/\pi = 0.95493$
$p(6) = -0.0208 < 0 < 0.0557 = p(7)$ ✓ (decay-rate zero between 6 and 7)
$\Omega_{19} = 0.51614$; $\Omega_{217} = 2.33490 \times 10^{-120}$
$p(19) = 0.55351 < 0.57236 < p(20) = 0.57914$ ✓ (continuous crossing $d_1^\star \approx 19.7308$)
$p(217) = 1.77101 < 1.77245 < p(218) = 1.77331$ ✓ (continuous crossing $d_2^\star \approx 217.6267$)
$I_0 = 0.51614 \times 2.33490 \times 10^{-120} = 1.2051 \times 10^{-120}$ ✓
$9/\pi^2 = 0.91189$ ✓
$I = 0.91189 \times 1.2051 \times 10^{-120} = 1.0990 \times 10^{-120}$ ✓
$\pi e^{2\sqrt{\pi}}(2\sqrt{\pi} - 1)/\ln 10 = 120.27$ ✓
$R_{\text{eff}}(5) = 1/\sqrt{8} = 0.354$; $R_{\text{eff}}(19) = 1/\sqrt{22} = 0.213$; $R_{\text{eff}}(217) = 1/\sqrt{220} = 0.067$

$\Omega_4/V_5 = 5$ ✓ ; $\Omega_6/V_7 = 7$ ✓
Structural stability verification:

K	$d_0(K)$	$d_1(K)$	$d_2(K)$
1.50	5	11	91
$\sqrt{\pi} = 1.77$	7	19	217
2.00	8	32	437
2.50	13	79	1856

Equilibria exist for all K; thresholds shift by orders of magnitude. ✓