Part 0 — Supplement

Inter-Layer Coupling and the Independent-Step Correction

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This supplement extends Paper 0 by computing the second-order correction to the cascade's independent-step approximation. The slicing recurrence treats each compactification step as independent: the propagator through n steps is the product of n individual lapses. This is exact for volumes (the telescoping identity $V_D/V_4 = \prod N(d)$ holds by the Gamma function). It is not exact for quantities that depend on the overlap structure of consecutive layer integrands. The correction is determined by the eigenvalue deficit of the Gram correlation matrix of the integrands $f_d(x) = (1 - x^2)^{d/2}$—a Beta function identity with no free parameters.

15 Inter-Layer Coupling

15.1 The Gram matrix of cascade layer integrands

Definition 15.1 (Gram matrix)**.** *For a set of cascade layers $\{d_1, d_2, \ldots, d_n\}$, the Gram matrix is*

$$G_{ij} = \langle f_{d_i}, f_{d_j} \rangle = \int_{-1}^{1} (1 - x^2)^{(d_i+d_j)/2}\, dx.$$

Theorem 15.2 (Gram matrix as Beta function)**.**

$$G_{ij} = B\Big(\tfrac{1}{2},\, \tfrac{d_i+d_j}{2} + 1\Big) = \sqrt{\pi}\, \frac{\Gamma\Big(\frac{d_i+d_j}{2} + 1\Big)}{\Gamma\Big(\frac{d_i+d_j}{2} + \frac{3}{2}\Big)}.$$

Proof. Substituting $t = x^2$ in the integral: $\int_{-1}^{1} (1 - x^2)^{\alpha}\, dx = B(1/2, \alpha + 1)$ with $\alpha = (d_i + d_j)/2$. The Beta function identity $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ with $a = 1/2$ gives the stated form. □

Definition 15.3 (Correlation matrix)**.** *The normalised Gram matrix (correlation matrix) is*

$$C_{ij} = \frac{G_{ij}}{\sqrt{G_{ii}\, G_{jj}}} = \frac{B\Big(\frac{1}{2},\, \frac{d_i+d_j}{2} + 1\Big)}{\sqrt{B\big(\frac{1}{2},\, d_i + 1\big)\; B\big(\frac{1}{2},\, d_j + 1\big)}}.$$

$C_{ii} = 1$ for all i, $\operatorname{tr} C = n$, and $0 < C_{ij} < 1$ for $i \neq j$.

15.2 The adjacent-layer overlap deficit

Theorem 15.4 (Overlap deficit)**.** *For adjacent layers d and $d+1$:*

$$1 - C_{d,d+1}^2 = 1 - \frac{B\left(\frac{1}{2},\, d+\frac{3}{2}\right)^2}{B\left(\frac{1}{2},\, d+1\right)\, B\left(\frac{1}{2},\, d+2\right)}.$$

This quantity is strictly positive for all $d \geq 1$ and satisfies:

$$1 - C_{d,d+1}^2 \sim \frac{1}{8\,d^2} \quad (d \to \infty).$$

Numerically: $1 - C_{5,6}^2 = 0.00319$, $1 - C_{11,12}^2 = 0.00083$, $1 - C_{19,20}^2 = 0.00030$.

Proof. Positivity: $C_{d,d+1} < 1$ by Cauchy–Schwarz (the integrands f_d and f_{d+1} are not proportional for $d \neq d+1$). Asymptotics: using Stirling, $B(1/2, \alpha+1) \approx \sqrt{2\pi/(2\alpha+1)}$. The ratio $C_{d,d+1}^2$ involves three Beta functions at arguments $d+3/2$, $d+1$, and $d+2$; expanding to second order in $1/d$ gives $1 - C^2 = 1/(8d^2) + O(1/d^3)$. □

Remark. The overlap deficit $1 - C_{d,d+1}^2$ is the geometric content at step d that is *not* shared with step $d+1$. It measures the imperfect folding: the fraction by which consecutive layer integrands fail to be collinear in L^2. Its positivity is a consequence of $R_{\text{eff}}(d) = 1/\sqrt{d+3} > 0$ (Theorem 4.2): the compactification at each step is asymptotic, never complete.

15.3 The eigenvalue structure of the correlation matrix

Theorem 15.5 (Near-collinearity)**.** *For a path of n consecutive layers $\{d_0, d_0+1, \ldots, d_0+n-1\}$, the correlation matrix C has one dominant eigenvalue λ_1 satisfying*

$$\frac{\lambda_1}{n} = 1 - \epsilon(n, d_0), \qquad \epsilon > 0,$$

and $(n-1)$ subdominant eigenvalues summing to $n\,\epsilon$. The deficit ϵ depends on both n (the number of layers) and d_0 (where in the cascade the path begins).

Proof. The entries C_{ij} are smooth functions of $|d_i - d_j|$ that decrease from $C_{ii} = 1$ towards a minimum at $|d_i - d_j| = n-1$. For the path $d = 5, \ldots, 12$ ($n = 8$): the minimum entry is $C_{5,12} = 0.9622$, far from zero. The matrix is therefore close to the rank-1 matrix $\mathbf{1}\mathbf{1}^T/n$, whose sole nonzero eigenvalue is n. The deficit ϵ is determined by the off-diagonal decay rate of C_{ij}, which is controlled by the Beta function asymptotics of Theorem 15.4. □

Numerical eigenvalue structure for $d = 5, \ldots, 12$ ($n = 8$):

Eigenvalue	Value	Fraction of trace
λ_1	7.9347263	99.184%
λ_2	0.0646528	0.808%
λ_3	0.0006152	0.008%
λ_4–λ_8	$< 10^{-4}$	$< 0.001\%$
$\epsilon = 1 - \lambda_1/n$	0.008159 = 0.816%	

The cascade layers are 99.2% collinear. The remaining 0.8% is the inter-layer coupling.

15.4 Path dependence of the eigenvalue deficit

The deficit $\epsilon(n, d_0)$ depends on where in the cascade the path lies. Lower d (closer to the observer, larger R_{eff}, less complete folding) gives larger deficits.

Path	n	$\epsilon = 1 - \lambda_1/n$	Needed correction
$d = 5, \dots, 12$	8	0.00816	1.73% (α_s)
$d = 6, \dots, 13$	8	0.00657	1.75% (m_τ/m_μ)
$d = 10, \dots, 17$	8	0.00332	—
$d = 14, \dots, 21$	8	0.00200	0.13% (m_μ/m_e)

At fixed n, ϵ decreases with d_0 because the overlap deficit per step (Theorem 15.4) decreases as $1/d^2$.

15.5 The independent-step correction

The cascade's leading-order predictions treat each compactification step as independent. A quantity that depends on the cascade's geometry over a path of n layers receives a correction from the inter-layer coupling. The eigenvalue deficit ϵ identifies the source, sign, and magnitude of this correction. The exact correction factor is observable-dependent.

Definition 15.6 (Independent-step correction)**.** *For a cascade quantity Q whose leading-order value Q_0 depends on the cascade potential over a path $\{d_0, \dots, d_0 + n - 1\}$, the corrected value is*

$$Q = Q_0 \times (1 + k_Q\,\epsilon)\,, \qquad \epsilon = 1 - \frac{\lambda_1(C)}{n}, \tag{1}$$

where $\lambda_1(C)$ is the largest eigenvalue of the $n \times n$ correlation matrix and k_Q is a coupling coefficient that depends on how the observable projects onto the subdominant eigenstructure of C.

Remark (What is derived and what is not). The eigenvalue deficit ϵ is derived: it follows from first-order perturbation theory applied to the deficit matrix $D = J - C$, where J is the all-ones matrix (Section 15.6). Its sign ($\epsilon > 0$), its path dependence (decreasing with d_0), and its scaling ($\epsilon \propto 1/d^2$ per step) are theorems about the Beta function. The coupling coefficient k_Q is not derived. It depends on how the observable couples to the subdominant eigenvectors of C, which varies from observable to observable. Computing k_Q from first principles is the most important open problem in this supplement.

Remark (What the correction is not). The correction is not to the Gamma function, which is exact. It is not to the slicing recurrence, which is exact (the telescoping identity $V_D/V_4 = \prod N(d)$ holds by $\Gamma(a+1) = a\Gamma(a)$). It is to the *independent-step approximation*: the assumption that a quantity depending on n cascade layers is the product of n individually exact factors. The layers are individually exact but not independent—their integrands overlap in L^2, and the eigenvalue deficit ϵ quantifies the overlap.

15.6 The perturbation theory for ϵ

Theorem 15.7 (Eigenvalue deficit from perturbation theory)**.**

$$\epsilon = 1 - \frac{\lambda_1}{n} = \frac{2}{n^2} \sum_{i<j} (1 - C_{ij})\,,$$

where C_{ij} are the entries of the correlation matrix.

Proof. Write $C = J - D$ where $J_{ij} = 1$ for all i, j and $D_{ij} = 1 - C_{ij} \geq 0$, with $D_{ii} = 0$. The matrix J has eigenvalue n with eigenvector $u = (1, \ldots, 1)/\sqrt{n}$ and eigenvalue 0 with multiplicity $n - 1$. By first-order perturbation theory:

$$\lambda_1(C) = n - \langle u|D|u\rangle = n - \frac{1}{n}\sum_{i,j} D_{ij} = n - \frac{2}{n}\sum_{i<j}(1 - C_{ij}).$$

Dividing by n and rearranging gives the stated formula. □

Verification: for $d = 5, \ldots, 12$ ($n = 8$), the perturbative formula gives $\epsilon = 0.008173$; the exact eigenvalue gives $\epsilon = 0.008159$. Agreement: 0.17%. The formula is verified to comparable precision for all paths tested.

15.7 Numerical evidence for the correction

Three quantities with unambiguous cascade paths test the correction. The coupling coefficient k_Q is fitted independently for each.

Observable	Path	n	ϵ	k_Q	Leading	Corrected	Observed
α_s	d=5–12	8	0.00816	2.11	0.1159	0.1179	0.1179
ℓ_A	d=5–12	8	0.00816	1.66	297.6	301.6	301.6
m_τ/m_μ	d=6–13	8	0.00657	2.67	16.53	16.82	16.82

All three residuals vanish when k_Q is fitted individually. The coupling coefficients cluster between 1.7 and 2.7. Their variation reflects the different ways each observable couples to the subdominant eigenstructure.

Remark (The approximate universality of $k_Q \approx 2$). Setting $k_Q = 2$ for all three quantities gives residuals of −0.09% (α_s), +0.27% (ℓ_A), and −0.43% (m_τ/m_μ)—sub-half-percent in every case, and a significant improvement over the uncorrected deviations of −1.70%, −1.34%, and −1.72%. The value $k_Q = 2$ is the best integer approximation (RMS residual 0.30%, compared to 0.87% for $k = 1$ and 0.76% for $k = 3$). The optimal least-squares value across all three is $k = 2.08$.

15.8 Sign structure of the correction

The correction $k_Q\epsilon$ is always positive ($\epsilon > 0$, $k_Q > 0$), so it increases the predicted value. This produces the observed sign structure across all cascade predictions:

- *Descent-dependent quantities* (those traversing multiple cascade layers) have negative leading-order deviations, because the independent-step approximation underestimates the propagated geometric content. The correction $+k_Q\epsilon$ closes the gap.
- *Geometric quantities* (single-level ratios, topological constants) have positive leading-order deviations, because they do not traverse multiple layers and receive no inter-layer correction. Their deviations reflect the subleading structure of the Gamma function at a single level, which is a separate (and smaller) effect.

The clean sign separation—negative for descent, positive for geometric—is not an empirical observation. It is a consequence of the Gram eigenvalue deficit being positive: $\epsilon > 0$ because $C_{ij} < 1$ for $i \neq j$ (Cauchy–Schwarz), so the deficit matrix D has positive entries and $\langle u|D|u\rangle > 0$.

15.9 The correction as compactification residual

Theorem 4.2 establishes that each compactification step retains a residual $R_{\text{eff}}(d) = 1/\sqrt{d+3}$. The overlap deficit (Theorem 15.4) scales as $1 - C_{d,d+1}^2 \sim 1/(8d^2) \sim R_{\text{eff}}^4/8$. The eigenvalue deficit ϵ is the cumulative effect of these per-step residuals over the path.

The independent-step approximation corresponds to the limit $R_{\text{eff}} \to 0$ (complete compactification). The correction $k_Q\epsilon$ is the leading-order cost of $R_{\text{eff}} > 0$—the imperfect folding of the cascade's own geometry.

15.10 The first-order correction: a Beta function identity

Theorem 15.8 (Adjacent overlap deficit correction)**.** *For a descent-dependent observable traversing path $\{d_0, \ldots, d_0 + n - 1\}$:*

$$\frac{\delta Q}{Q_0} = \sum_{k=0}^{n-2} \bigl(1 - C_{d_k, d_{k+1}}^2\bigr), \tag{2}$$

where each $C_{d,d+1}$ is the adjacent-layer correlation (Theorem 15.4). This is a sum of Beta function ratios. No free parameter enters.

Proof. The observable propagates through n layers. The independent-step approximation treats each layer as perfectly collinear with the next ($C_{d,d+1} = 1$). The first-order departure from collinearity at each step is the squared overlap deficit $1 - C_{d,d+1}^2$, which measures the fraction of the integrand at step d that is orthogonal to step $d+1$ (Theorem 15.4). For a multiplicative propagator, the leading correction is the sum of per-step deficits. □

Observable	Path	n	$\sum(1-C^2)$	Leading	Corrected	Observed	Dev
α_s	$d = 5$–12	8	0.01186	0.1159	0.1173	0.1179	−0.5%
m_τ/m_μ	$d = 6$–13	8	0.00938	16.53	16.69	16.82	−0.8%
m_μ/m_e	$d = 14$–21	8	0.00272	206.50	207.06	206.77	+0.1%
m_τ (MeV)	$d = 5$–12	8	0.01186	1755	1776	1777	−0.1%
v (GeV)	$d = 5$–12	8	0.01186	240.8	243.7	246.2	−1.0%
Ω_m^{Bott}	$d = 5$–217	213	0.02108	0.3115	0.3181	0.315	+1.0%

The first-order correction reduces descent deviations from $\sim 2\%$ to sub-1% for all tested quantities. The correction for m_μ/m_e is small (0.27%) because its path lies at higher d where the per-step deficit scales as $1/(8d^2)$.

Corollary 15.9 (Derived coupling coefficient)**.** *The first-order coupling coefficient is*

$$k_Q^{(1)} = \frac{\sum_{\text{adj}} (1 - C_{d,d+1}^2)}{\epsilon}, \tag{3}$$

where the numerator is the adjacent overlap deficit sum (Theorem 15.8) and the denominator is the eigenvalue deficit (Theorem 15.7). Both are sums of Beta function ratios. No free parameter enters.

Proof. $k_Q^{(1)}$ is defined by equating the first-order correction $\delta Q/Q_0 = \sum(1-C^2)$ with the parametrisation $k_Q\,\epsilon$. The ratio is $\sum(1-C^2)/\epsilon$, with both quantities derived. □

Path	n	$k_Q^{(1)}$	Residual after $k_Q^{(1)}$
$d = 5, \ldots, 12$	8	1.454	sub-0.6%
$d = 6, \ldots, 13$	8	1.429	sub-0.8%
$d = 14, \ldots, 21$	8	1.362	sub-0.2%

Remark (Status of the second-order correction). The fitted k_Q values (2.11, 1.66, 2.67) exceed $k_Q^{(1)}$ for some observables and fall below it for others. The variation is observable-dependent: it reflects how each quantity's functional form responds to the universal geometric correction $\delta\Phi = \sum(1-C^2)$. For direct exponentials ($\alpha_s = \alpha_{\mathrm{GUT}} \times e^{\Phi}$), the response is close to 1. For integrated quantities (ℓ_A, involving the comoving distance integral), the response is smoothed below 1. For quantities on higher-d paths (m_μ/m_e), the first-order over-corrects and the effective $k_Q < k_Q^{(1)}$. The second-order factor is a property of the observable's formula, not of the cascade's geometry.

15.11 What this section proves

1. The Gram matrix of cascade layer integrands is a matrix of Beta function values (Theorem 15.2).
2. Adjacent layers have a strictly positive overlap deficit $1 - C^2_{d,d+1} > 0$, scaling as $1/(8d^2)$ (Theorem 15.4).
3. The correlation matrix has one dominant eigenvalue $\lambda_1 \approx n(1-\epsilon)$ with $\epsilon > 0$ (Theorem 15.5).
4. The eigenvalue deficit is derived from first-order perturbation theory as a sum of Beta function deficits (Theorem 15.7).
5. The sign of the correction (positive) explains the clean sign separation between descent and geometric predictions.
6. The first-order coupling coefficient $k_Q^{(1)} = \sum(1-C^2)/\epsilon$ is derived (Corollary 15.9): a ratio of Beta function sums with no free parameter. It reduces all descent deviations to sub-1%.
7. The residual variation between observables (the second-order factor) is a property of each observable's functional form, not of the cascade's geometry. It is observable-dependent and open.
8. Every quantity in this section—including $k_Q^{(1)}$—is a Gamma function identity. No physics enters.

15.12 Open questions

1. *Derive the observable-dependent response factor.* The first-order correction $\delta Q/Q_0 = \sum(1-C^2)$ (Theorem 15.8) and its coupling coefficient $k_Q^{(1)} = \sum(1-C^2)/\epsilon$ (Corollary 15.9) are derived. They reduce descent deviations from $\sim$2% to sub-1% for all tested quantities. The residual variation is observable-dependent: for direct exponentials (α_s) the first-order under-corrects by $\sim$0.5%; for integrated quantities (ℓ_A) it over-corrects slightly; for high-d paths (m_μ/m_e) it over-corrects by $\sim$0.1%. The second-order factor is a response function $R_Q = k_Q/k_Q^{(1)}$ that depends on how each

observable's formula converts the geometric correction $\delta\Phi$ into a prediction shift $\delta Q/Q$. Deriving R_Q for each observable is a calculus problem (differentiate the observable's formula with respect to the cascade potential), not a cascade geometry problem.

2. *Analytic formula for* $\epsilon(n, d_0)$. The deficit is computable numerically from the Gram eigenvalue. The perturbation theory (Theorem 15.7) gives an exact sum; an analytic closed form in terms of n, d_0, and the Beta function would extend the correction to all cascade paths without numerical eigenvalue computation.
3. *Higher-order corrections.* The subdominant eigenvalues $\lambda_3, \lambda_4, \ldots$ contribute at third order and beyond. For the $d = 5, \ldots, 12$ path, $\lambda_2/n = 0.81\%$ and $\lambda_3/n = 0.008\%$: the third-order correction is negligible. For longer paths, it may not be.
4. *Application to mixed quantities.* The correction applies cleanly to quantities with a single unambiguous cascade path (α_s, ℓ_A, mass ratios). Quantities depending on multiple interacting paths (absolute masses, the electroweak scale) require decomposing the observable into its cascade-path components.

Remark (Status of the correction). The first-order correction (Theorem 15.8) is derived: it is a sum of Beta function ratios with no free parameters, reducing descent deviations from sub-2% to sub-1%. The second-order (observable-dependent) correction is the remaining open problem at the level of the Gram-matrix framework of this supplement. With both orders derived, descent-dependent predictions would close to sub-0.1%. Open Question 1 is the calculation that separates the two at the Gram-matrix level.

Remark (Superseded for α_s, m_τ/m_μ, and m_τ). The Gram-matrix framework above is a general structural account of how inter-layer coupling corrects the independent-step approximation. For three specific Standard Model precision observables—$\alpha_s(M_Z)$, m_τ/m_μ, and the τ absolute mass—a sharper structural identity has subsequently been identified in Part IVb. Both deviations in these observables are closed *at experimental precision* by cascade potential shifts of the form $\delta\Phi = \alpha(d^*)/\chi$, where d^* is a distinguished dimension from this paper's four-dimension tower (Theorem 7.1) and $\chi = \chi(S^{2n}) = 2$ is the Euler characteristic of the SU(2) sphere:

- $\delta\Phi_{\mathrm{U}(1)} = \alpha(14)/\chi$ closes α_s and m_τ/m_μ together (Part IVb, Theorem 4.3);
- $\delta\Phi_{\mathrm{phase}} = \alpha(19)/\chi$ closes the τ absolute mass at the phase-transition threshold $d_1 = 19$ (Part IVb, Theorem 4.4).

These identities supersede the Gram-matrix corrections for the three observables named: the universal $k_Q \approx 2$ fit used for α_s and m_τ/m_μ in Section 15.7 above is replaced by a parameter-free $\alpha(d^*)/\chi$ closed form that works at experimental precision rather than sub-1%. The m_μ/m_e match through the Gram-matrix first-order correction remains the current best account for that observable. Open Question 1 (the observable-dependent second-order R_Q factor) is accordingly narrowed: the three highlighted observables no longer require it, but the remaining descent-dependent quantities (v, m_W, Ω_m, ℓ_A, other absolute masses) may be closed either by further members of the $\alpha(d^*)/\chi$ family or by a subleading Gram-matrix response, to be determined.